LABORATOIRE JEAN KUNTZMANN
MATHÉMATIQUES APPLIQUÉES - INFORMATIQUE

cnrs

Communauté UNIVERSITÉ Grenoble Alpes

Analyse, synthèse et super-résolution de structures orientées dans les images

Modélisation de la diffusion d'information dans les réseaux sociaux

25 avril 2018

Kévin Polisano

Data Institute
Univ. Grenoble Alpes

LIG

Présentation

Contexte du post-doctorat

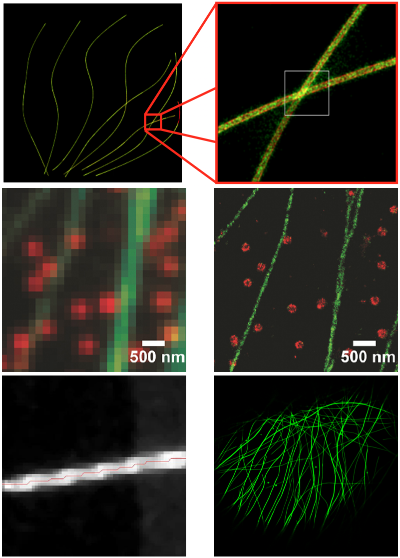
- **L**aboratoire d'**I**nformatique de **G**renoble (LIG)
- Équipe AMA (d**A**ta analysis, **M**odeling, m**A**chine learning)
- Collaborateurs :
  Eric Gaussier (LIG, AMA)
  Adeline Leclerc-Samson (LJK, SVH)
  Jean-Marc Francony (LSS, Régulations)
- Titre : "Modélisation multi-échelle de la diffusion d'information et de la dynamique d'opinion"
- Projet du Data Institute

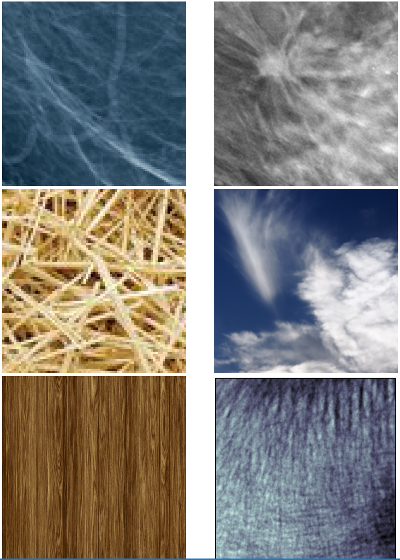
Présentation

Contexte du doctorat

- **L**aboratoire **J**ean **K**untzmann (LJK)
- Équipe CVGI (**C**alcul des **V**ariations, **G**éométrie, **I**mage)
- Encadrants : Valérie Perrier (Director)
Marianne Clausel (Co-supervisor)
Laurent Condat (Co-supervisor)
- Titre : "Modélisation de textures anisotropes par la transformée en ondelettes monogènes et super-résolution de lignes 2D", *soutenue le 12 décembre 2017*.
- ATER à l'Université Grenoble Alpes (2017–2018)

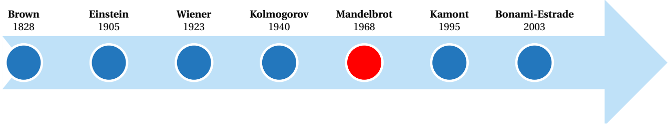
Motivations





Plan de l'exposé

Du brownien aux champs aléatoires anisotropes



Autosimilarité

$\{X(t)\}_{t\in T}$ est dit autosimilaire de paramètre H si $\forall\lambda \in \mathbb{R}$

$$\{X(\lambda t)\}_{t\in T} \overset{(fdd)}{=} \lambda^H \{X(t)\}_{t\in T}$$

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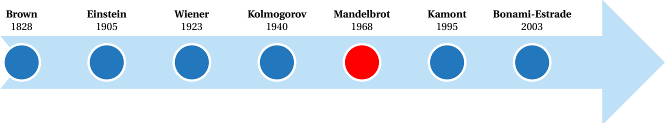

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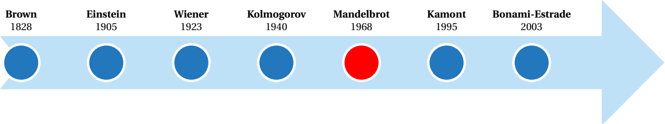

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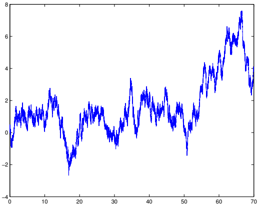

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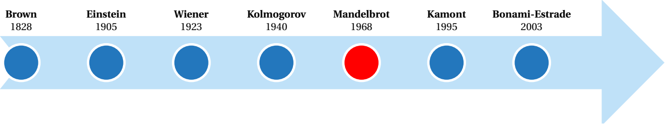
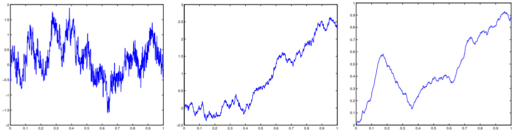


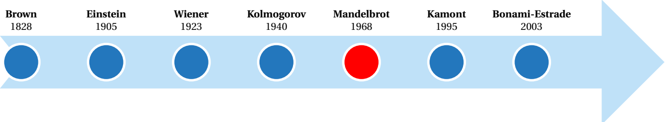
- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t - s|^{2H} \Rightarrow$ ~~accr. indépendants~~

Mouvement brownien fractionnaire B^H



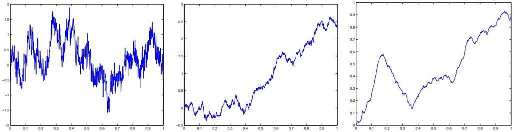
$H = 0.2$ $\quad$ $H = 0.5$ $\quad$ $H = 0.8$

Du brownien aux champs aléatoires anisotropes



- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ accr. stationnaires

Mouvement brownien fractionnaire B^H (FBM)



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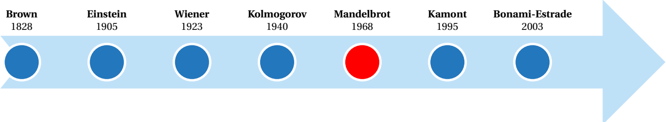
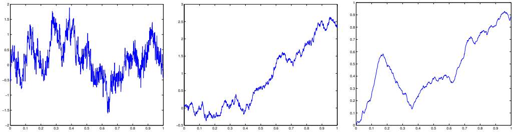
Du brownien aux champs aléatoires anisotropes



- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ accr. stationnaires
- $\mathbf{R}(t,s) = \mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$

Mouvement brownien fractionnaire B^H (FBM)



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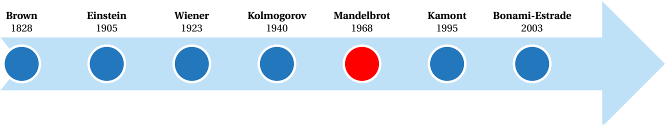

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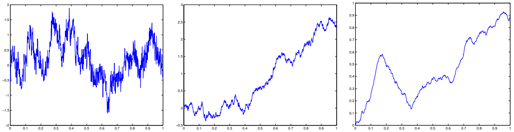
- $\mathbf{R}(t,s) = \mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$
- $B^H(t) = \frac{1}{c_H} \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{j}t\xi}-1}{|\xi|^{H+1/2}} \widehat{\mathbf{W}}(\xi) \Rightarrow$ représentation spectrale

Mouvement brownien fractionnaire B^H (FBM)



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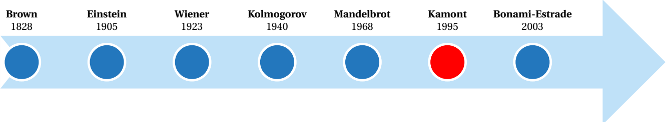
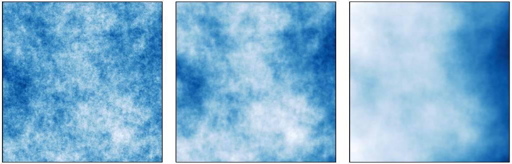
Du brownien aux champs aléatoires anisotropes



- $\mathbb{E}\left[(B^H(\boldsymbol{x}) - B^H(\boldsymbol{y}))^2\right] = \|\boldsymbol{x}-\boldsymbol{y}\|^{2H}$, $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^2$
- $\mathbf{R}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\left(\|\boldsymbol{x}\|^{2H} + \|\boldsymbol{y}\|^{2H} - \|\boldsymbol{x}-\boldsymbol{y}\|^{2H}\right)$
- $B^H(\boldsymbol{x}) = \frac{1}{C_H}\int_{\mathbb{R}^2} \frac{\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle}-1}{\|\boldsymbol{\xi}\|^{H+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$

Champ brownien fractionnaire B^H (FBF)



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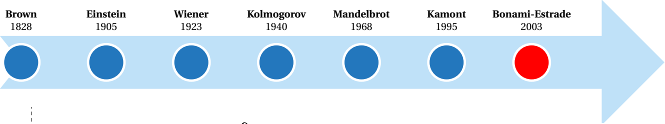
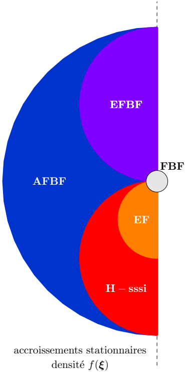
Modèle de Bonami-Estrade





$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{\xi}) \, \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

- $f^{1/2}(\boldsymbol{\xi}) = \dfrac{C}{\|\boldsymbol{\xi}\|^{H+1}}$ (FBF)
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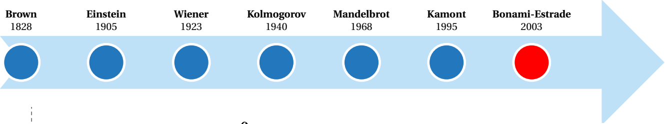

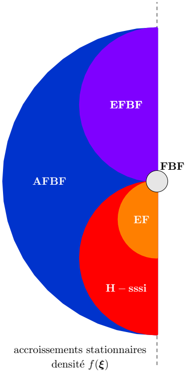

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1) f^{1/2}(\boldsymbol{\xi})\, \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

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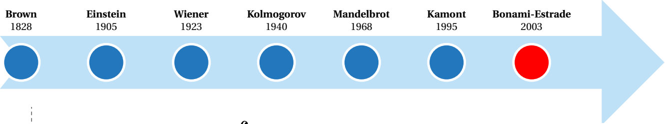

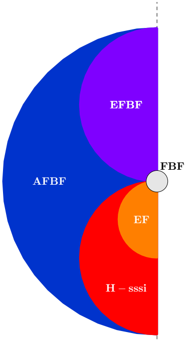

accroissements stationnaires
densité $f(\boldsymbol{\xi})$

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{\xi}) \, \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

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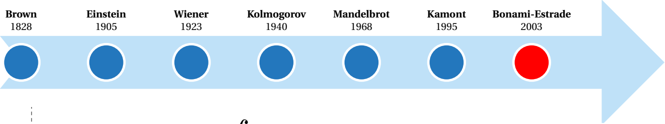

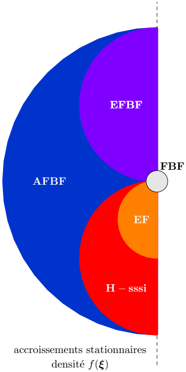

$$X(x) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle x, \xi \rangle} - 1) f^{1/2}(\xi)\, \widehat{\mathbf{W}}(\mathrm{d}\xi)$$

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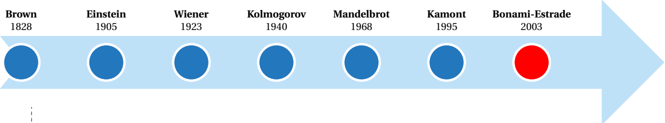
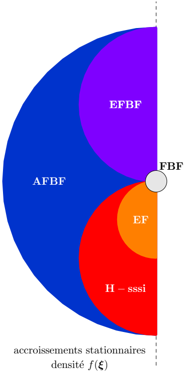
Modèle de Bonami-Estrade





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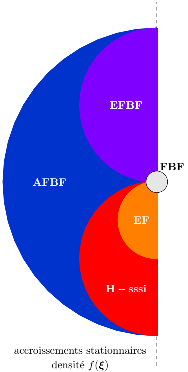
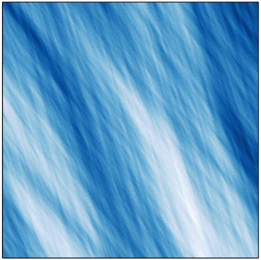
  - $$C(\boldsymbol{\xi}) = \mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha_0) \quad \text{(EF)}$$

Modèle de Bonami-Estrade

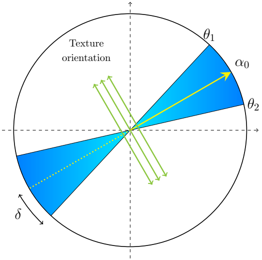
$$X(x) = \int_{\mathbb{R}^2} \left(e^{j\langle x, \xi\rangle} - 1\right) \frac{\mathbb{1}_{[-\delta,\delta]}(\arg \xi - \alpha_0)}{\|\xi\|^{H+1}} \widehat{W}(d\xi)$$



Champ élémentaire (EF) $[H = 0.5,\ \alpha_0 = \pi/6]$



$\delta = 3.10^{-1}$



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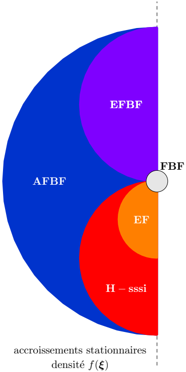

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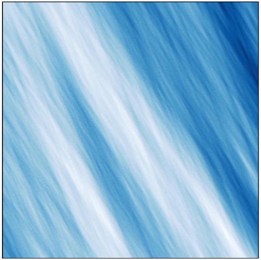

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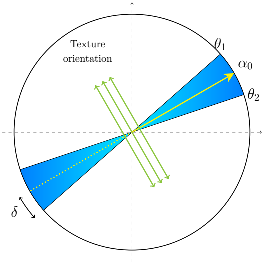

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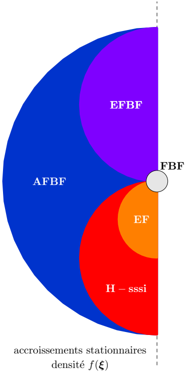

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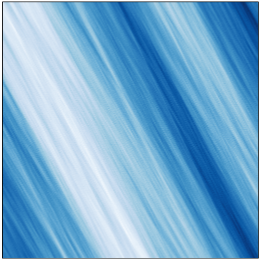

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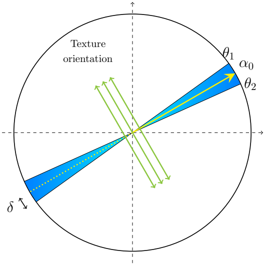

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État de l'art des champs anisotropes

- Drap brownien fractionnaire (FBS) (Kamont, 1995), (Léger et Pontier, 1999), (Ayache et coll., 2002)
- Champs H-sssi (Benassi et coll., 1997)
- Modèle de Bonami et Estrade (Bonami et Estrade, 2003)
- Champs gaussiens à autosimilarité matricielle (OSGRF) (Schertzer et Lovejoy, 1985), (Biermé et coll., 2007)
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$\Rightarrow$ aucune classe de champs à anisotropie locale contrôlée

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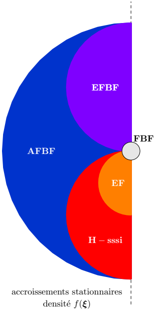
$\Rightarrow$ aucune classe de champs à anisotropie locale contrôlée

$\Rightarrow$ contribution : deux nouvelles classes de ce type
le (GAFBF) et le (WAFBF)

Des champs H-sssi aux GAFBF

- Modèle de Bonami-Estrade :

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{\xi})\, \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$



H-sssi

$$f^{1/2}(\boldsymbol{\xi}) = \frac{C(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{H+1}}$$

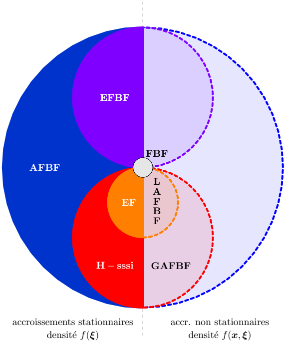
EF

$$C(\boldsymbol{\xi}) = \mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha_0)$$

Modèle à orientation et régularité locales prescrites

- Notre modèle :

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1\right) f^{1/2}(\boldsymbol{x}, \boldsymbol{\xi}) \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$



GAFBF

$$f^{1/2}(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{C(\boldsymbol{x}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x})+1}}$$

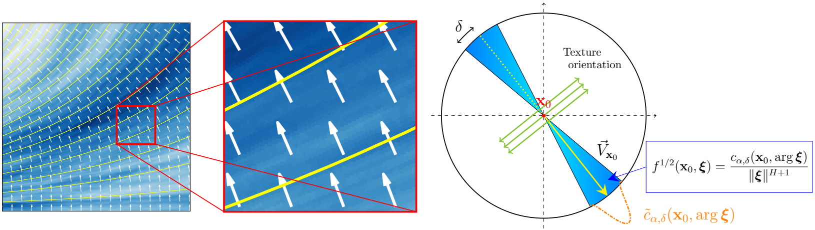
LAFBF

$$C(\boldsymbol{x}, \boldsymbol{\xi}) = \mathbb{1}_{[-\delta(\boldsymbol{x}), \delta(\boldsymbol{x})]}(\arg \boldsymbol{\xi} - \alpha(\boldsymbol{x}))$$

Modèle à orientation locale prescrite

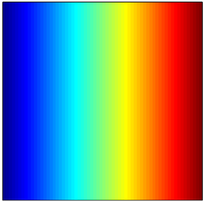
$$B^H_{\alpha,\delta}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle} - 1\right) \frac{\mathbb{1}_{[-\delta,\delta]}(\arg\boldsymbol{\xi} - \alpha(\boldsymbol{x}))}{\|\boldsymbol{\xi}\|^{H+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

Champ élémentaire localisé (LAFBF) $[H = 0.8,\ \alpha(x_1, x_2) = -\pi/2 + x_1]$

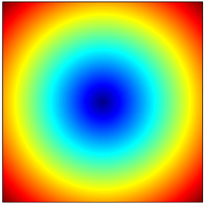


(a) (b) (c)

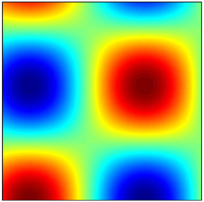
Simulation du LAFBF à h variable (krigeage)



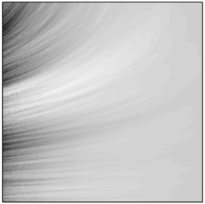
h linéaire



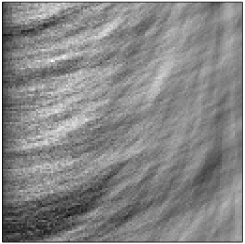
h radiale



h sinusoïdale



Simulation du LAFBF à h variable

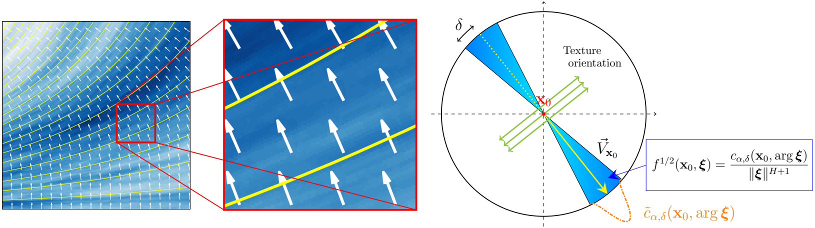


- Variation linéaire des orientations $\alpha(\boldsymbol{x})$ selon (Ox)
- Variation linéaire de la directionnalité $\delta(\boldsymbol{x})$ selon (Ox)
- Variation linéaire de la régularité $h(\boldsymbol{x})$ selon (Ox)

Le champ tangent : outil d'analyse et de synthèse

Le champ tangent comme outil d'analyse (Lévy-Vehel, 1995), (Falconer, 2002) :

$$\left\{ \lim_{\rho \to 0} \frac{X(\mathbf{x_0} + \rho \mathbf{x}) - X(\mathbf{x_0})}{\rho^{h(\mathbf{x}_0)}} \right\}_{\mathbf{x} \in \mathbb{R}^2} \stackrel{d}{=} \left\{ Y_{\mathbf{x}_0}(\mathbf{x}) \right\}_{\mathbf{x} \in \mathbb{R}^2}$$



(a) (b) (c)

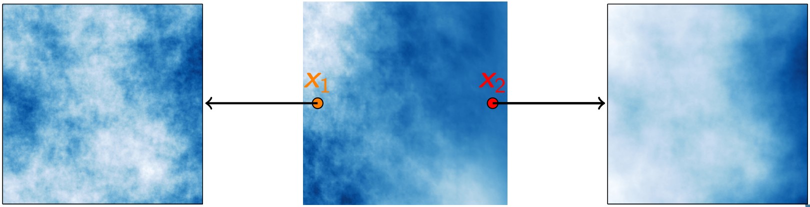
Le champ tangent : outil d'analyse et de synthèse

Champ brownien multifractionnaire B^h (MBF) (Peltier, Vehel, 1995)

Le MBF se comporte localement comme un FBF

$$\left\{ \lim_{\rho \to 0} \frac{B^h(\mathbf{x_0} + \rho \mathbf{x}) - B^h(\mathbf{x_0})}{\rho^{h(\mathbf{x}_0)}} \right\}_{\mathbf{x} \in \mathbb{R}^2} \stackrel{d}{=} \left\{ B^{h(\mathbf{x}_0)}(\mathbf{x}) \right\}_{\mathbf{x} \in \mathbb{R}^2}$$

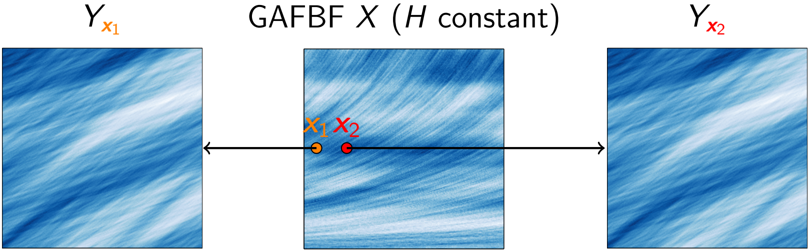
FBF B^H, $H \equiv h(x_1)$ MBF $B^h(\mathbf{x})$ FBF B^H, $H \equiv h(x_2)$



Champ tangent du GAFBF

Soit X le GAFBF défini par

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1\right) \frac{C(\boldsymbol{x}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x})+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$



Champ tangent du GAFBF

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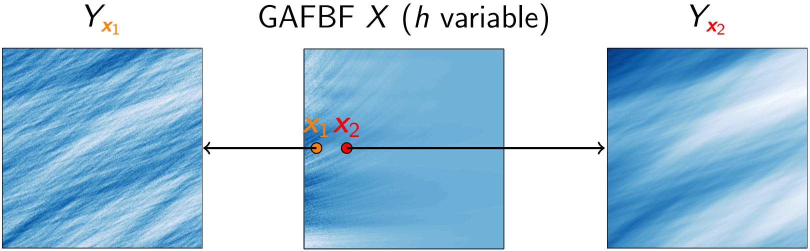


Hypothèses du GAFBF

Hypothèses ($\mathcal{H}$)

- h est β–höldérienne, telle que $a = \inf_{\boldsymbol{x}\in\mathbb{R}^2} h(\boldsymbol{x}) > 0$, $b = \sup_{\boldsymbol{x}\in\mathbb{R}^2} h(\boldsymbol{x})$ et $b < \beta \leq 1$.
- $(\boldsymbol{x},\boldsymbol{\xi}) \mapsto C(\boldsymbol{x},\boldsymbol{\xi})$ est bornée $C(\boldsymbol{x},\boldsymbol{\xi}) \leqslant M, \forall(\boldsymbol{x},\boldsymbol{\xi})$.
- $\boldsymbol{\xi} \mapsto C(\boldsymbol{x},\boldsymbol{\xi})$ est paire $C(\boldsymbol{x},-\boldsymbol{\xi}) = C(\boldsymbol{x},\boldsymbol{\xi})$.
- $\boldsymbol{\xi} \mapsto C(\boldsymbol{x},\boldsymbol{\xi})$ homogène $C(\boldsymbol{x},\rho\boldsymbol{\xi}) = C(\boldsymbol{x},\boldsymbol{\xi}), \forall\rho$.
- $\boldsymbol{x} \mapsto C(\boldsymbol{x},\boldsymbol{\xi})$ est continue et $\exists\eta$, $b \leq \eta \leq 1$, t.q $\forall\boldsymbol{x}$

$$\sup_{\boldsymbol{z}\in B(\mathbf{0},1)} \|\boldsymbol{z}\|^{-2\eta} \int_{\mathbb{S}^1} \left[C(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{\Theta}) - C(\boldsymbol{x},\boldsymbol{\Theta})\right]^2 \mathrm{d}\boldsymbol{\Theta} \leq A_{\boldsymbol{x}} < \infty$$

Champ tangent du GAFBF

Soit X le GAFBF défini par

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1\right) \frac{C(\boldsymbol{x}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x})+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

Théorème (Polisano et coll., 2017)

Si X vérifie les hypothèses $(\mathcal{H})$, alors X admet en tout point $\boldsymbol{x_0} \in \mathbb{R}^2$ un champ tangent $Y_{\boldsymbol{x_0}}$ donné par la représentation :

$$\begin{aligned} Y_{\boldsymbol{x_0}}(\boldsymbol{x}) &= \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{x_0}, \boldsymbol{\xi}) \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi}) \,, \\ &= \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C(\boldsymbol{x_0}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x_0})+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi}) \,. \end{aligned}$$

Champ tangent du GAFBF

Soit X le GAFBF défini par

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1\right) \frac{C(\boldsymbol{x}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x})+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

Théorème (Polisano et coll., 2017)

Si X vérifie les hypothèses $(\mathcal{H})$, alors X admet en tout point $\boldsymbol{x}_0 \in \mathbb{R}^2$ un champ tangent $Y_{\boldsymbol{x}_0}$ donné par la représentation :

$$Y_{\boldsymbol{x}_0}(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1) f^{1/2}(\boldsymbol{x}_0, \boldsymbol{\xi}) \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi}) \,,$$

$$\text{champ H-sssi} = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1) \frac{C_{\boldsymbol{x}_0}(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x}_0)+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi}) \,.$$

Orientation locale d'une fonction déterministe

Opérateur de gradient

L'opérateur de gradient $\nabla : f \mapsto (\partial_{x_1} f, \partial_{x_2} f)$, avec la notation $\partial_{x_1} f : \boldsymbol{x} = (x_1, x_2) \mapsto \frac{\partial f}{\partial x_1}(\boldsymbol{x})$, est défini en Fourier par :

$$\widehat{\partial_{x_1} f}(\boldsymbol{\omega}) = -\mathrm{j}\omega_1 \widehat{f}(\boldsymbol{\omega}), \quad \widehat{\partial_{x_2} f}(\boldsymbol{\omega}) = -\mathrm{j}\omega_2 \widehat{f}(\boldsymbol{\omega})$$

$\Rightarrow$ Orientation :

$$\boldsymbol{n}(\boldsymbol{x}) = \frac{\nabla f(\boldsymbol{x})}{\|\nabla f(\boldsymbol{x})\|}$$

Orientation locale d'une fonction déterministe

Transformée de Riesz et signal monogène (Felsberg, 2001)

L'opérateur de Riesz $\boldsymbol{\mathcal{R}} : f \mapsto (\mathcal{R}_1 f, \mathcal{R}_2 f)$ est défini par :

$$\widehat{\mathcal{R}_1 f}(\boldsymbol{\omega}) = -\mathrm{j}\frac{\omega_1}{\|\boldsymbol{\omega}\|}\widehat{f}(\boldsymbol{\omega}), \quad \widehat{\mathcal{R}_2 f}(\boldsymbol{\omega}) = -\mathrm{j}\frac{\omega_2}{\|\boldsymbol{\omega}\|}\widehat{f}(\boldsymbol{\omega})$$

$\Rightarrow$ Orientation :

$$\boldsymbol{n}(\boldsymbol{x}) = \frac{\boldsymbol{\mathcal{R}} f(\boldsymbol{x})}{\|\boldsymbol{\mathcal{R}} f(\boldsymbol{x})\|}$$

Orientation pour un champ H-sssi

Coefficients d'ondelettes monogènes d'un champ H-sssi X

$$c_{i,\boldsymbol{k}}^{(\ell)}(X) = \langle X, \mathcal{R}_\ell \psi_{i,\boldsymbol{k}} \rangle = \int_{\mathbb{R}^2} \widehat{\mathcal{R}_\ell \psi_{i,\boldsymbol{k}}}(\boldsymbol{\xi}) C(\boldsymbol{\xi}) \|\boldsymbol{\xi}\|^{-H-1} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

Théorème (Polisano et al., 2017)

Définissons $c_{i,\boldsymbol{k}}^{(\mathcal{R})}(X) = (c_{i,\boldsymbol{k}}^{(1)}(X), c_{i,\boldsymbol{k}}^{(2)}(X))^\mathsf{T}$, alors :

$$\mathbb{E}[c_{i,\boldsymbol{k}}^{(\mathcal{R})}(X) c_{i,\boldsymbol{k}}^{(\mathcal{R})}(X)^*] \propto 2^{-2i(H+1)} \mathbf{J}_X \ ,$$

où $\mathbf{J}_X$ est appelé le tenseur de structure de X défini par :

$$[\mathbf{J}_X]_{\ell_1 \ell_2} = \int_{\boldsymbol{\Theta} \in \mathbb{S}^1} \Theta_{\ell_1} \Theta_{\ell_2} \ C(\boldsymbol{\Theta})^2 \, \mathrm{d}\boldsymbol{\Theta}, \quad \ell_1, \ell_2 \in \{1, 2\} \ .$$

Orientation pour un champ H-sssi

Définition (Orientation et indice de cohérence du H-sssi X)

- L'orientation $\vec{\boldsymbol{n}}_X$ de X est le vecteur propre unitaire associé à la plus grande des valeurs propres λ_1, λ_2 de $\mathbf{J}_X$
- L'indice de cohérence de X est défini par

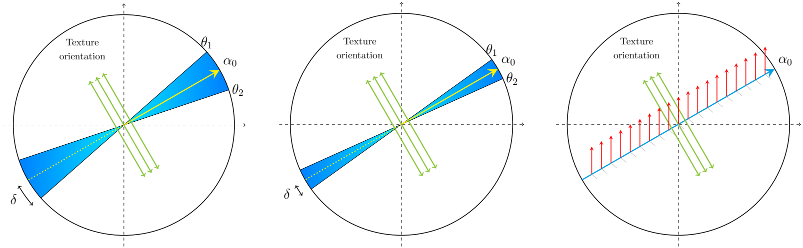
$$\chi = \frac{|\lambda_2 - \lambda_1|}{\lambda_1 + \lambda_2}$$

Orientations d'un champ élémentaire

Exemple (Orientation d'un EF)

$X = X_{\alpha_0,\delta}$ avec $C(\mathbf{\Theta}) = \mathbb{1}_{[-\delta,\delta]}(\arg \mathbf{\Theta} - \alpha_0)$

$$\vec{\boldsymbol{n}}_X = \begin{pmatrix} \cos \alpha_0 \\ \sin \alpha_0 \end{pmatrix}, \quad \chi = \frac{\sin(2\delta)}{2\delta}$$



Orientation d'un champ gaussien localisable

Champ gaussien localisable

Un champ aléatoire $X = \{X(\boldsymbol{x}),\, \boldsymbol{x} \in \mathbb{R}^2\}$ est dit localisable, si il admet un champ tangent en tout point $\boldsymbol{x} \in \mathbb{R}^2$.

Références : (Lévy-Véhel, 1995), (Benassi et coll., 1997), (Falconer, 2002).

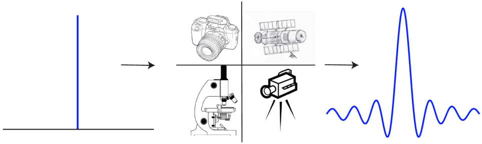
Définition (Orientation locale d'un champ gaussien localisable)

L' orientation locale $\vec{\boldsymbol{n}}_X(\boldsymbol{x}_0)$ du champ gaussien localisable X au point $\boldsymbol{x_0}$ est l'orientation de son champ tangent $Y_{\boldsymbol{x_0}}$ H-sssi :

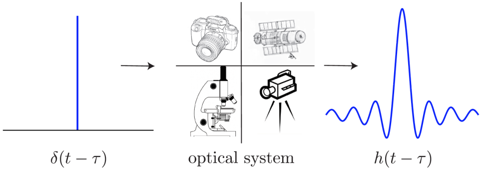
$$\vec{\boldsymbol{n}}_X(\boldsymbol{x}_0) \equiv \vec{\boldsymbol{n}}_{Y_{\boldsymbol{x}_0}}$$

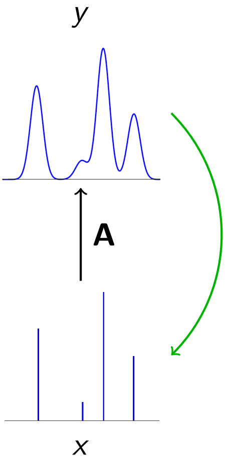
Plan de l'exposé

Diffraction et limite de Rayleigh



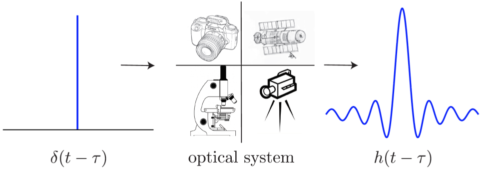
$\delta(t-\tau)$ optical system $h(t-\tau)$

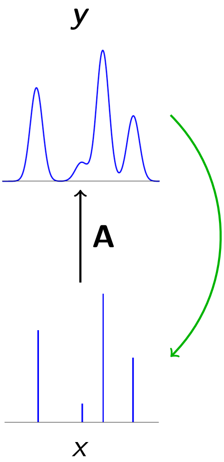




$$x = \sum_{i=1}^{K} c_i \delta_{t_i}, \quad c_i \geq 0, \quad t_i \geq 0$$

Super-resolution d'impulsions 1-D

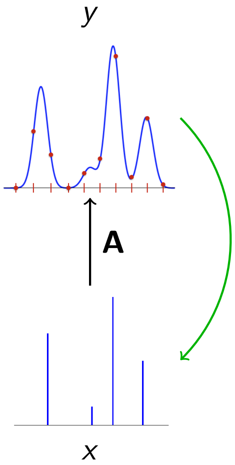




$$y(t) = \sum_{i=1}^{K} c_i h(t - t_i), \quad c_i \geq 0, \quad t_i \geq 0$$

Échantillonnage discret sur la grille

$$\boldsymbol{y} = y(\tau_k), \quad \tau_k = k\Delta/N$$



Échantillonnage discret sur la grille

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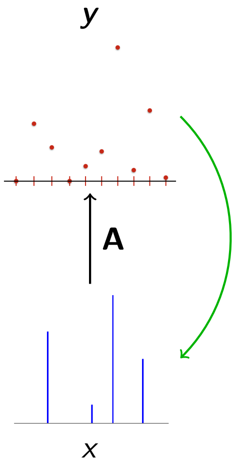
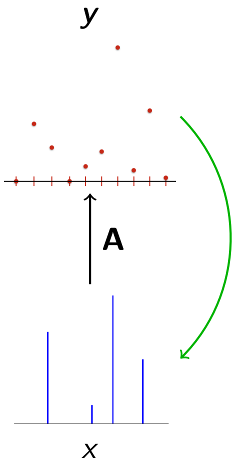


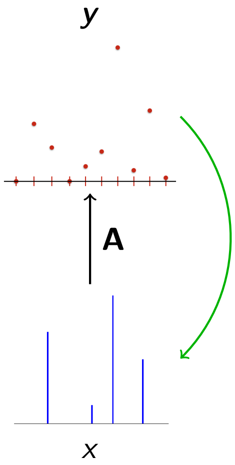
Reconstruction parcimonieuse sur la grille

$$\min_{\boldsymbol{c}\in\mathbb{R}^K} \frac{1}{2}\left\|\boldsymbol{y}-\mathbf{A}\boldsymbol{c}\right\|_2^2 + \lambda\left\|\boldsymbol{c}\right\|_0$$



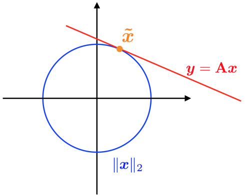
Reconstruction parcimonieuse convexe sur la grille

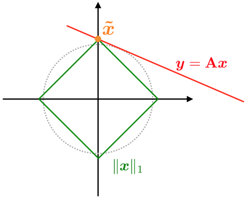
$$\min_{\boldsymbol{c} \in \mathbb{R}^K} \frac{1}{2} \|\boldsymbol{y} - \mathbf{A}\boldsymbol{c}\|_2^2 + \lambda \|\boldsymbol{c}\|_1$$

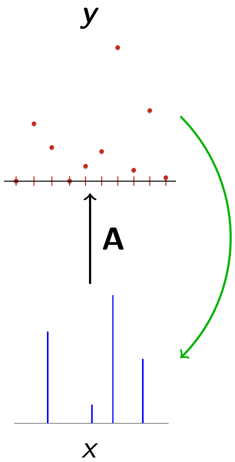


Reconstruction parcimonieuse convexe sur la grille

$$\min_{\boldsymbol{c}\in\mathbb{R}^K} \frac{1}{2}\|\boldsymbol{y}-\mathbf{A}\boldsymbol{c}\|_2^2+\lambda\|\boldsymbol{c}\|_1$$



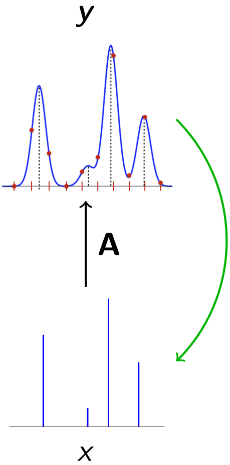




Super-resolution d'impulsions 1-D sur la grille

$$\min_{\boldsymbol{c}\in\mathbb{R}^K} \frac{1}{2}\|\boldsymbol{y}-\mathbf{A}\boldsymbol{c}\|_2^2+\lambda\|\boldsymbol{c}\|_1$$

$$\boldsymbol{y}=y(\tau_k),\qquad \tau_k=k\Delta/N\longrightarrow\tilde{x}_k$$



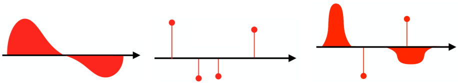
Super-resolution 1-D affranchie de la grille

$$x = \sum_{i=1}^{K} c_i \delta_{t_i}, \quad c_i \geq 0, \quad t_i \geq 0$$

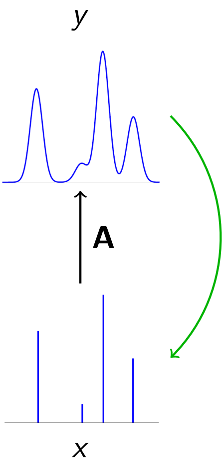
Minimization (convex regularization)

$$\arg\min_{\mu} \frac{1}{2} \|y - \mathbf{A}\mu\|^2 + \lambda \|\mu\|_{\mathrm{TV}}$$

Référence : (Candès, Fernandez-Granda, 2012)

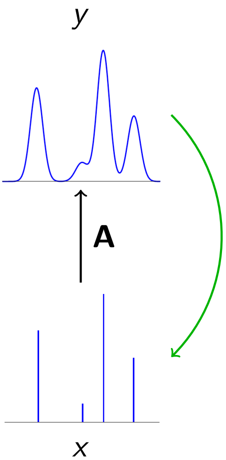


$\|\mu\|_{\mathrm{TV}} = \int |f|$ $\|x\|_{\mathrm{TV}} = \|\boldsymbol{c}\|_1$



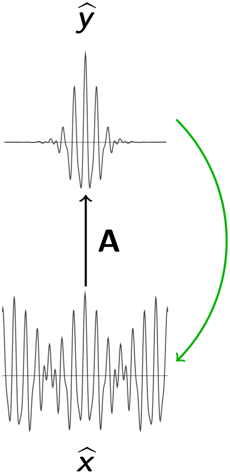
Super-resolution 1-D affranchie de la grille

$$(\mathcal{F}x)(\omega) = \sum_{i=1}^{K} c_i \mathrm{e}^{\mathrm{j}2\pi f_i \omega}, \quad c_i \geq 0, \quad t_i \geq 0$$



Super-resolution 1-D affranchie de la grille

$$(\mathcal{F}x)(\omega) = \sum_{i=1}^{K} c_i \mathrm{e}^{\mathrm{j}2\pi f_i \omega}, \quad c_i \geq 0, \quad t_i \geq 0$$

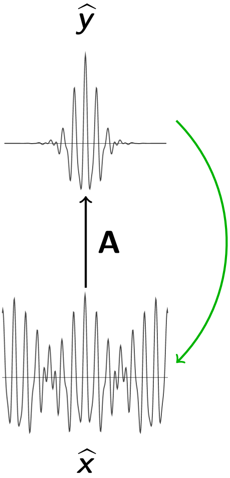


Super-resolution 1-D affranchie de la grille

$$\widehat{x} = \sum_{i=1}^{K} c_i \mathbf{a}(f_i), \quad c_i \geq 0, \quad \mathbf{a}(f_i) \in \mathcal{A}$$

$$\mathcal{A} = \left\{ \boldsymbol{a}(f) \in \mathbb{C}^N \right\}, \quad [\boldsymbol{a}(f)]_n = \mathrm{e}^{\mathrm{j}2\pi f n}$$

$$\|\widehat{x}\|_{\mathcal{A}} = \inf \left\{ \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} : \widehat{x} = \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} \mathbf{a} \right\}$$



Minimisation (régularisation convexe)

$$\arg\min_{\widehat{x}} \frac{1}{2} \|\widehat{y} - \mathbf{A}\widehat{x}\|^2 + \lambda \|\widehat{x}\|_{\mathcal{A}}$$

Référence : (Tang, Bhaskar, Recht et coll., 2013)

Super-resolution 1-D affranchie de la grille

$$\widehat{x} = \sum_{i=1}^{K} c_i \mathbf{a}(f_i), \quad c_i \geq 0, \quad \mathbf{a}(f_i) \in \mathcal{A}$$

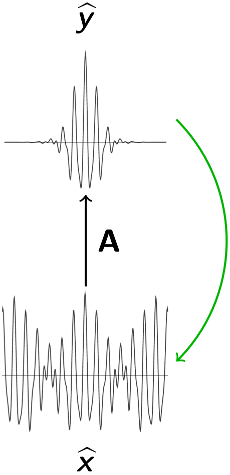
$$\mathcal{A} = \left\{ \boldsymbol{a}(f) \in \mathbb{C}^N \right\}, \quad [\boldsymbol{a}(f)]_n = \mathrm{e}^{\mathrm{j}2\pi f n}$$

$$\|\widehat{\boldsymbol{x}}\|_{\mathcal{A}} = \min_{\boldsymbol{q} \in \mathbb{C}^N} \left\{ q_0 : \begin{pmatrix} \mathbf{T}_N(\boldsymbol{q}) & \boldsymbol{t}_m^\sharp \\ \boldsymbol{t}_m^{\sharp\,*} & q_0 \end{pmatrix} \succcurlyeq 0 \right\}$$

Minimisation (régularisation convexe)

$$\arg\min_{\widehat{\boldsymbol{x}}} \frac{1}{2} \|\widehat{\boldsymbol{y}} - \mathbf{A}\widehat{\boldsymbol{x}}\|^2 + \lambda \|\widehat{\boldsymbol{x}}\|_{\mathcal{A}}$$

Référence : (Tang, Bhaskar, Recht et coll., 2013)



Enhance it ! Vers une super-resolution 2-D

Problème inverse

Minimisation (attache aux données)

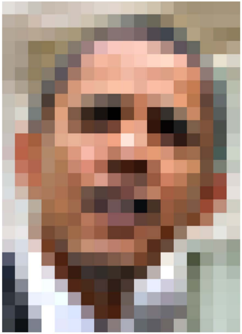
Problème mal posé :

$$\arg\min_{\mathbf{x}} \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2$$

Exemple (Opérateur de dégradation)

- $\mathbf{A}$ = sous-échantillonnage
- $\mathbf{A}$ = flou
- ...

Enhance it ! Vers une super-resolution 2-D



Enhance it ! Vers une super-resolution 2-D

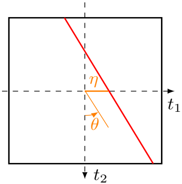
Problème inverse

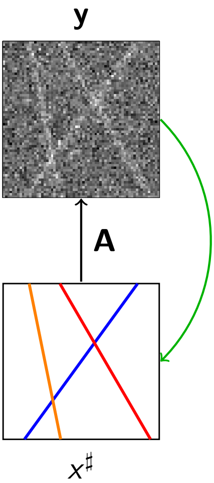
Minimisation (régularisation convexe)

$$\arg\min_{x} \frac{1}{2}\left\|\mathbf{y}-\mathbf{A}x\right\|^{2}+\lambda\left\|x\right\|_{\mathrm{TV}}$$

$$\mathbf{y}^{\sharp}[n_1,n_2] \overset{\mathbf{A}x^{\sharp}}{=} (x^{\sharp}*\phi)(n_1,n_2)+\epsilon$$

$$x^{\sharp}(t_1,t_2)=\sum_{k=1}^{K}\alpha_k\delta\big(\cos\theta_k(t_1-\eta_k)+\sin\theta_k\, t_2\big)$$



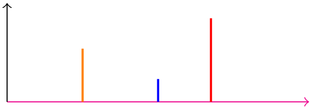


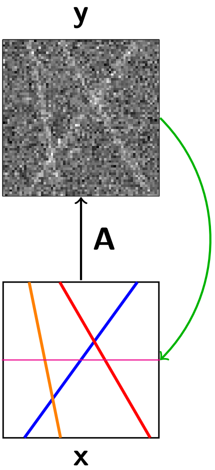
Problème inverse

Minimisation (régularisation convexe)

$$\arg\min_{x} \frac{1}{2} \left\|\mathbf{y} - \mathbf{A}x\right\|^2 + \lambda \left\|x\right\|_{\mathrm{TV}}$$

$$x^\sharp(t_1, t_2) = \sum_{k=1}^{K} \alpha_k \delta\big(\cos\theta_k(t_1 - \eta_k) + \sin\theta_k\, t_2\big)$$





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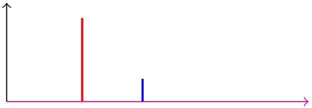
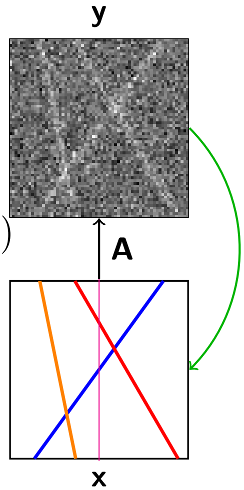


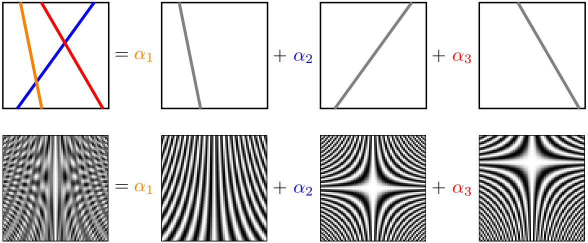


Modélisation des droites

$$\widehat{\mathbf{x}}^\sharp[m, n_2] = (\mathcal{F}_1 x^\sharp)[m, n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right) m}$$

$$c_k = \frac{\alpha_k}{\cos\theta_k} \geq 0$$

$$\widehat{\mathbf{y}} = \mathbf{A}\widehat{\mathbf{x}}^\sharp + \widehat{\epsilon}$$



Problème d'optimisation convexe

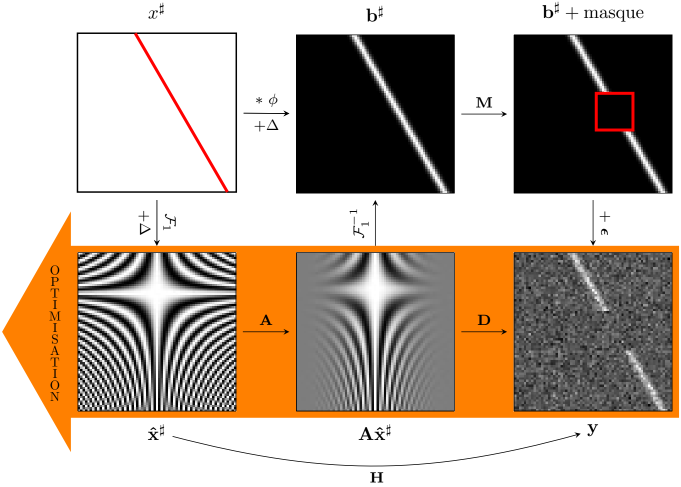
Proposition (Minimisation convexe)

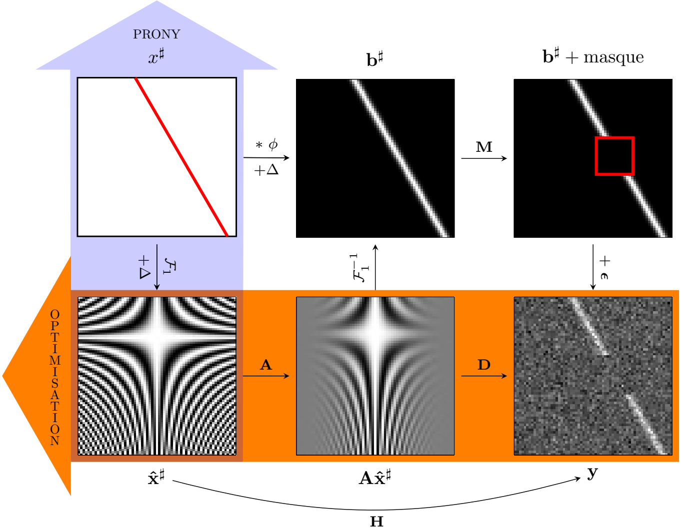
$$\tilde{\mathbf{x}} \in \underset{\widehat{\mathbf{x}}, \mathbf{q} \in \mathcal{X} \times \mathcal{Q}}{\arg\min} \frac{1}{2} \|\mathbf{A}\widehat{\mathbf{x}} - \widehat{\mathbf{y}}\|^2 ,$$

$$\text{sous contraintes} \quad \begin{cases} \forall n_2 = 0, \dots, H_S - 1, \ \forall m = 0, \dots, M , \\ \widehat{\mathbf{x}}[0, n_2] = \widehat{\mathbf{x}}[0, 0] \leqslant c , \\ \mathbf{q}[m, 0] \leqslant c , \\ \mathbf{T}'_{H_S}(\widehat{\mathbf{x}}[m, :], \mathbf{q}[m, :]) \succcurlyeq 0 , \\ \mathbf{T}_{M+1}(\widehat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 . \end{cases}$$

(Chambolle et Pock, 2010)

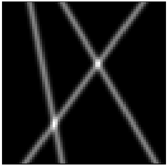
$$\tilde{\mathbf{X}} = \underset{\mathbf{X} \in \mathcal{H}}{\arg\min} \left\{ F(\mathbf{X}) + G(\mathbf{X}) + \sum_{i=0}^{Q-1} H_i(\mathrm{L}_i(\mathbf{X})) \right\}$$


$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$* \phi$
$+\Delta$
$\mathbf{M}$
$+\Delta$
$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
$+\epsilon$
OPTIMISATION
$\mathbf{A}$
$\mathbf{D}$
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
$\mathbf{H}$

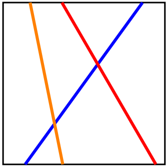

PRONY
$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$* \phi$
$+\Delta$
$\mathbf{M}$
$+\Delta$
$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
$+\epsilon$
OPTIMISATION
$\mathbf{A}$
$\mathbf{D}$
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
$\mathbf{H}$

Expériences numériques

- Débruitage et déconvolution



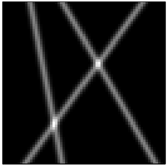
Exp. 1



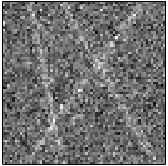
Détection

Expériences numériques

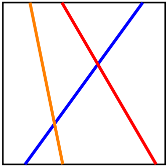
- Débruitage et déconvolution



Exp. 1



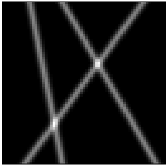
Exp. 2



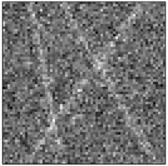
Détection

Expériences numériques

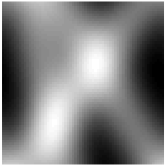
- Débruitage et déconvolution



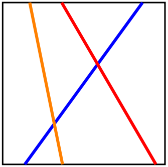
Exp. 1



Exp. 2



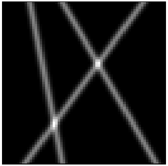
Exp 3.



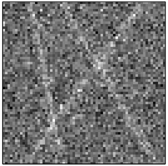
Détection

Expériences numériques

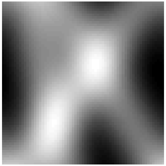
- Débruitage et déconvolution



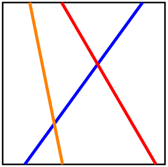
Exp. 1



Exp. 2



Exp 3.



Détection

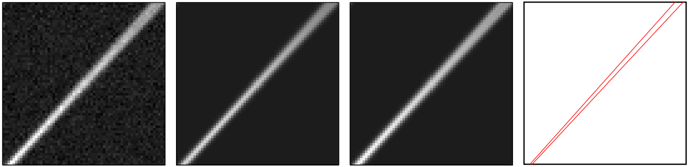
TABLE: Erreurs relatives de l'estimation des paramètres des lignes

	Expérience 1	Expérience 2	Expérience 3
Δ_θ/θ	$(10^{-7}, 3.10^{-6}, 7.10^{-7})$	$(10^{-2}, 6.10^{-2}, 9.10^{-2})$	$(6.10^{-7}, 9.10^{-5}, 8.10^{-6})$
Δ_α/α	$(10^{-7}, 10^{-7}, 10^{-7})$	$(10^{-2}, 9.10^{-2}, 2.10^{-1})$	$(4.10^{-5}, 2.10^{-5}, 2.10^{-5})$
Δ_η	$(4.10^{-6}, 7.10^{-6}, 7.10^{-6})$	$(5.10^{-2}, 4.10^{-2}, 3.10^{-2})$	$(5.10^{-5}, 10^{-4}, 3.10^{-4})$

Expériences numériques

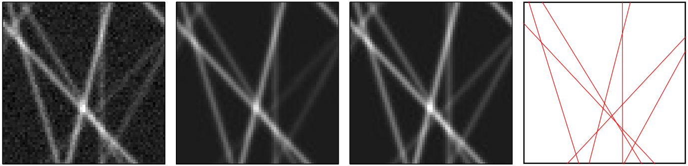
Expériences numériques

- Lignes proches



Bruitée — Débruitée — Sans bruit — Détection

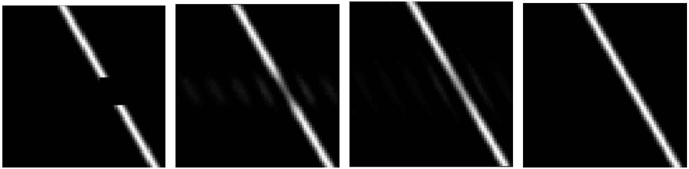
- Lignes multiples



Bruitée — Débruitée — Sans bruit — Détection

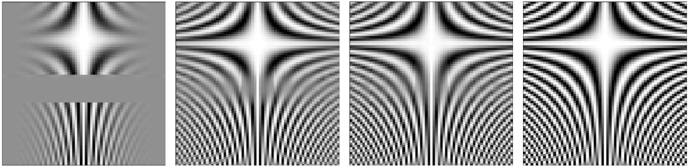
Expériences numériques

- Inpainting spatial



Masquage | iter $= 2000$ | iter $= 10000$ | iter $\to \infty$

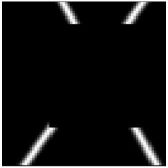
- Inpainting en Fourier

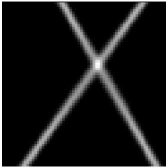


Masquage | iter $= 2000$ | iter $= 10000$ | iter $\to \infty$

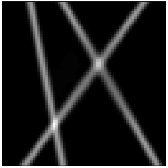
Expériences numériques

- Inpainting masquage important


Masquage

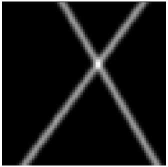

Inpainting


Masquage

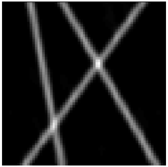

Inpainting

- Inpainting masquage aléatoire


Masquage


Inpainting


Masquage


Inpainting

Conclusion

- Deux nouveaux modèles de champs autosimilaires anisotropes à orientation et régularité locales prescrites
- Méthodes de synthèse efficaces
- Introduction d'une notion d'orientation locale pour les champs aléatoires
- Caractérisation de la loi des estimateurs statistiques de l'orientation
- Nouvelle méthode de super-résolution de lignes 2-D

Perpectives

- Amélioration et prolongation des méthodes :
  - Définition de la transformée de Riesz d'un champ
  - Test d'hypothèse de directionnalité d'une texture
  - Décomposition atomique 2-D et algorithme de Frank-Wolfe alternant étape convexe et non convexe
- Applications :
  - Tests d'orientation sur des images médicales
  - Super-résolution de *patchs* sur images de microscopie
- Perspectives à plus long terme :
  - Traiter le cas des orientations multiples
  - Super-résolution de courbes 2-D

Publications

Revues internationales avec actes et comité de lecture

K. Polisano, L. Condat, M. Clausel, V. Perrier, A convex approach to super-resolution and regularization of lines in images, *SIAM Journal on Imaging Sciences (SIIMS)*, 2018.

K. Polisano, M. Clausel, V. Perrier, L. Condat, Riesz-based orientation of localisable Gaussian fields, *under review in Applied and Computational Harmonic Analysis (ACHA)*, 2018.

K. Polisano, M. Clausel, V. Perrier, L. Condat, Simulation of oriented pattern with prescribed local orientation, *preprint*, 2018.

Conférences internationales avec actes et comité de lecture

K. Polisano, L. Condat, M. Clausel, and V. Perrier, Convex super-resolution detection of lines in images, *24th European Signal Processing Conference (EUSIPCO)*, 29 Aug. – 2 Sep, pp. 336-340, 2016.

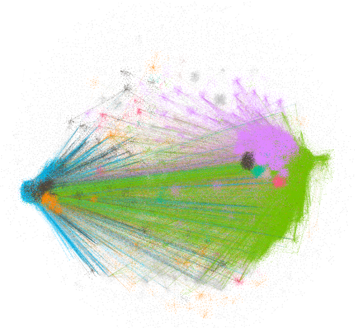
K. Polisano, M. Clausel, V. Perrier and L. Condat, Texture modeling by Gaussian fields with prescribed local orientation, *IEEE International Conference On Image Processing (ICIP)*, 27-30 Oct, pp. 6091-6095, 2014.

Plan de l'exposé

Post-doctorat

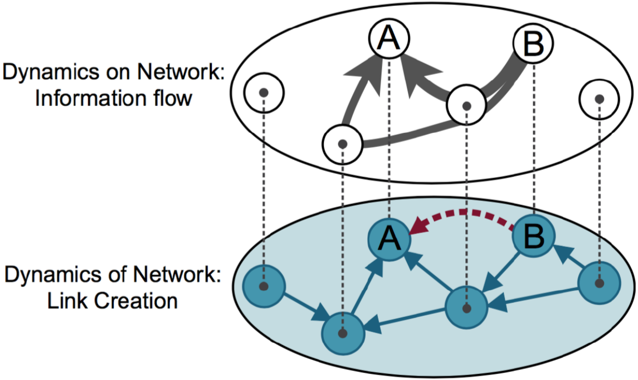
Modélisation multi-échelle de la diffusion d'information et de la dynamique d'opinion sur les réseaux sociaux

- 1 Base de données "GPA/PMA" collectée sur 1 an
- 1M de tweets
- 150 000 utilisateurs



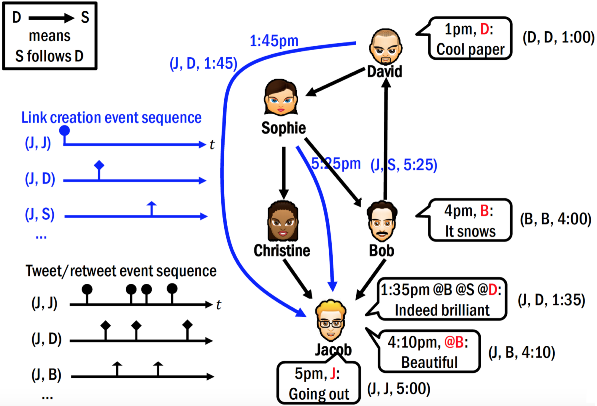
Post-doctorat

Évolution temporelle du réseau social



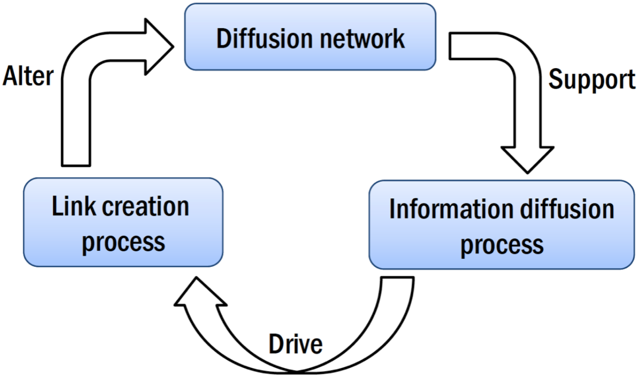
Post-doctorat

Exemple de construction dynamique du réseau social



Post-doctorat

Co-évolution dynamique



Post-doctorat

Modèle de co-évolution dynamique

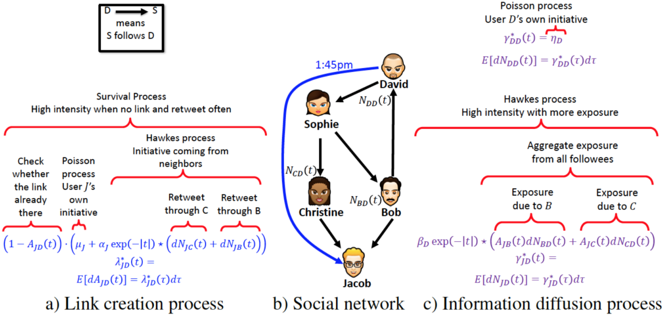
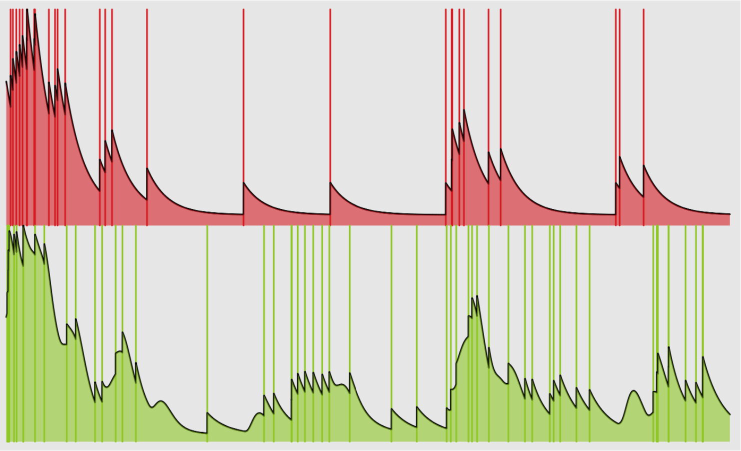


Figure 2.4: The breakdown of conditional intensity functions for 1) information diffusion process of Jacob retweeting posts originated from David $N_{JD}(t)$; 2) information diffusion process of David tweeting on his own initiative $N_{DD}(t)$; 3) link creation process of Jacob following David $A_{JD}(t)$

Post-doctorat

Hawkes multi-dimensionnel



Post-doctorat

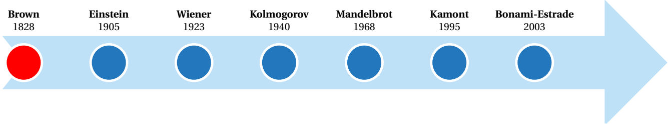
Perspectives d'amélioration

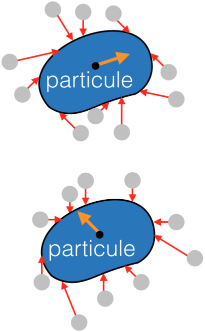
- Injecter de l'information sur les noeuds (profil utilisateur)
- Tenir compte du contenu des tweets (Bag of words, word embedding)
- Étudier le choix des noyaux, envisager des influences non-linéaires (deep learning)
- Proposer un modèle bidirectionnel à partir de processus de Hawkes sur des clusters plutôt que sur les noeuds individuels

Des questions ?

Merci de votre attention

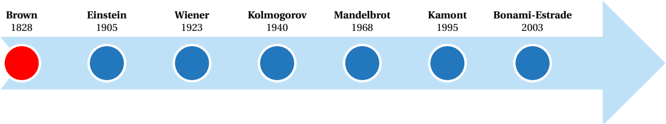
Du brownien aux champs aléatoires anisotropes



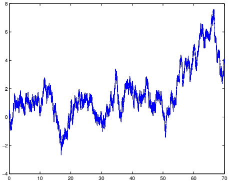


- déplacements indépendants
- distribution gaussienne
- trajectoire irrégulière

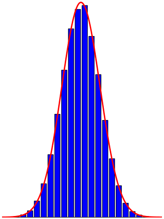
Du brownien aux champs aléatoires anisotropes



Déplacements indépendants

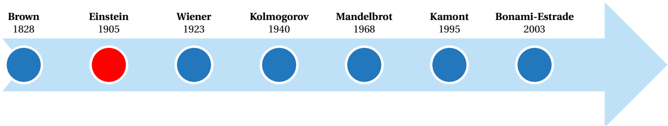


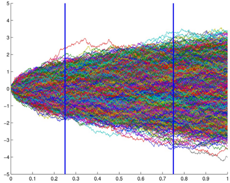
Trajectoire irrégulière

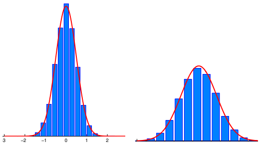


Distribution gaussienne

Du brownien aux champs aléatoires anisotropes

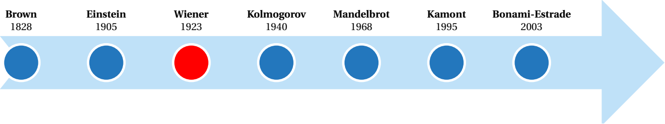






$$\overline{(\Delta x)^2} \propto t$$

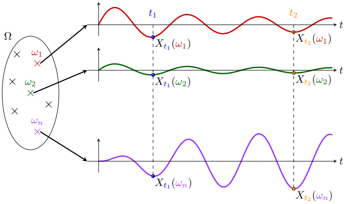
Du brownien aux champs aléatoires anisotropes



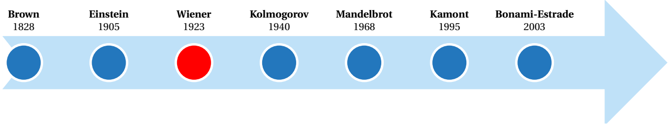
Mouvement brownien

- $(B_t)_t$ est à accroissements indépendants, $B_0 = 0$ p.s.
- $B_{t_i} - B_{t_j} \sim \mathcal{N}(0, t_i - t_j)$
- $(B_t)_t$ est à trajectoires p.s. continues.

$$\begin{array}{cccccc} X & : & T \times \Omega & \longrightarrow & E \\ & & (t,\omega) & \longmapsto & X(t,\omega) \end{array}$$



Du brownien aux champs aléatoires anisotropes



Isométrie $\mathbf{W}$: $(L^2, \langle f, g\rangle_{L^2}) \to (\mathcal{G}, \mathbb{E}[XY])$

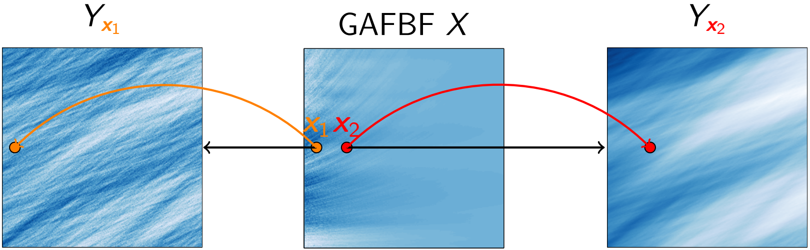
- $\mathbb{E}[\mathbf{W}(f)\mathbf{W}(g)] = \langle f, g\rangle_{L^2}$, $\quad \mathbf{W}(f) \sim \mathcal{N}(0, \|f\|_{L^2}^2)$
- $\forall t \in [0,1], \quad B_t \overset{\text{def}}{=} \mathbf{W}(\mathbb{1}_{[0,t]})$
  - $\mathbb{E}\left[(B_t - B_s)^2\right] = \left\|\mathbb{1}_{[0,t]} - \mathbb{1}_{[0,s]}\right\|_{L^2}^2 = \int \mathbb{1}_{[s,t]} = t - s$
  - $\mathbb{E}\left[(B_{t_i} - B_{t_i-1})(B_{t_j} - B_{t_j-1})\right] = \langle \mathbb{1}_{[t_{i-1},t_i]}, \mathbb{1}_{[t_{j-1},t_j]}\rangle_{L^2} = 0$

$$\text{intégrale de Wiener} = \int f(x)\mathbf{W}(\mathrm{d}x)$$

Synthèse du GAFBF par champ tangent

$$X(\boldsymbol{x}_0) \leftarrow Y_{\boldsymbol{x}_0}(\boldsymbol{x} = \boldsymbol{x}_0) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C_{\boldsymbol{x}_0}(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x}_0)+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

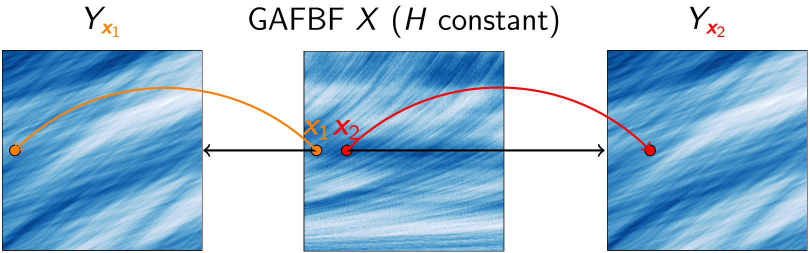
$\Rightarrow$ Nécessite de simuler autant de champs tangents qu'il y a de pixels dans l'image !



Synthèse du GAFBF inspirée de (Wood, 1994)

1. Simuler U GAFBF X^{H_u} à régularité constante $(H_u)_{1\leq u\leq U}$:

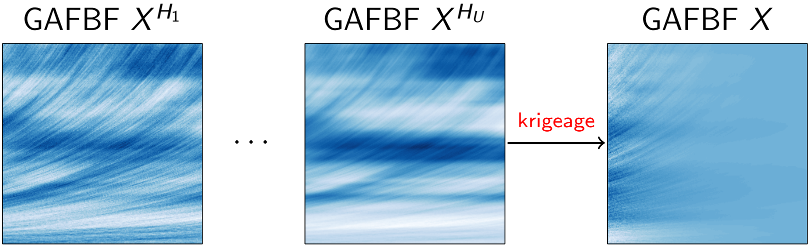
$$X^{H_u}(\boldsymbol{x}_0) \leftarrow Y_{\boldsymbol{x}_0}(\boldsymbol{x}=\boldsymbol{x}_0) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle}-1)\frac{C_{\boldsymbol{x}_0}(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{H_u+1}}\widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$



Synthèse du GAFBF inspirée de (Wood, 1994)

❷ Simuler le GAFBF à régularité variable par krigeage :

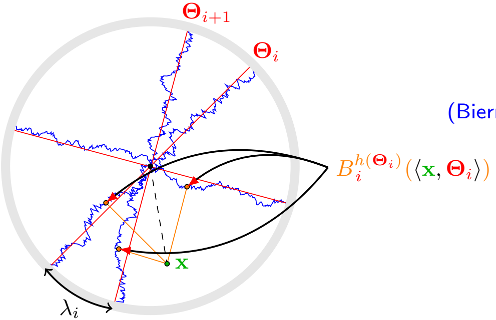
Interpolation spatiale des (X^{H_u}) à partir de la covariance



Synthèse du champ tangent (bandes tournantes)

$$Y_{\boldsymbol{x}_0}^{[n]}(\boldsymbol{x}) = \sum_{i=1}^{n} \omega_i(\boldsymbol{x}_0) B_i^H \left(\langle \boldsymbol{x}, \boldsymbol{\Theta}_i \rangle \right) \ ,$$

$$\omega_i(\boldsymbol{x}_0)^2 = \lambda_i \gamma(h(\boldsymbol{x}_0)) C_{\boldsymbol{x}_0}(\boldsymbol{\Theta}_i)$$



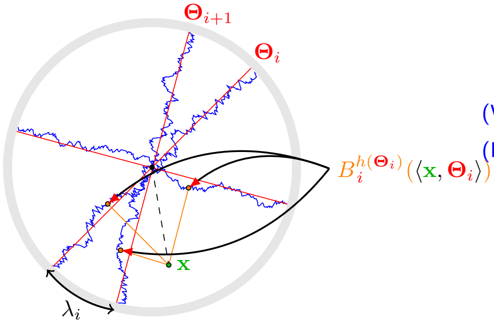
(Matheron, 19

(Biermé, Richard, Moisan, 20

Synthèse du champ tangent (bandes tournantes)

$$Y_{\boldsymbol{x}_0}^{[n]}(\boldsymbol{x}) = \sum_{i=1}^{n} \omega_i(\boldsymbol{x}_0) B_i^H(\langle \boldsymbol{x}, \boldsymbol{\Theta}_i \rangle) ,$$

$\Rightarrow$ Simuler n FBM B_i^H de complexité $O(\ell \log \ell)$



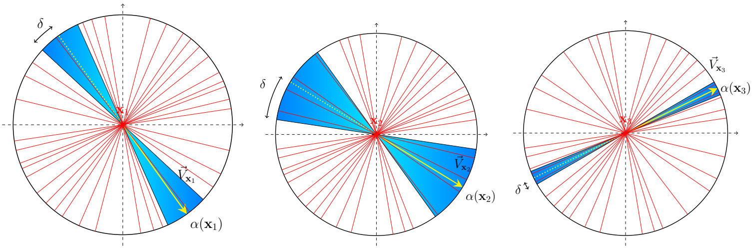
(Wood et coll., 1994)

(Perrin et coll., 2002)

Simulation du LAFBF à H constant

$$B^H_{\alpha,\delta}(\mathbf{x}_0) \leftarrow Y^{[n]}_{\mathbf{x}_0}(\mathbf{x} = \mathbf{x}_0) = \sum_{i=1}^{n} \omega_i(\mathbf{x}_0) B_i^H\left(\langle \mathbf{x}_0, \mathbf{\Theta}_i \rangle\right) ,$$

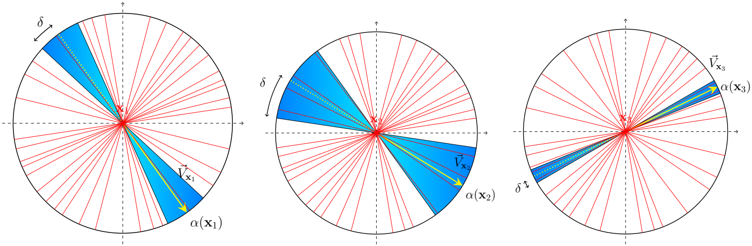
$$\omega_i(\mathbf{x}_0)^2 \propto C_{\mathbf{x}_0}(\mathbf{\Theta}_i) = \mathbb{1}_{[-\delta(\mathbf{x}_0),\delta(\mathbf{x}_0)]}(\arg \mathbf{\Theta}_i - \alpha(\mathbf{x}_0))$$



Simulation du LAFBF à H constant

$$B^H_{\alpha,\delta}(\mathbf{x}_0) \leftarrow Y^{[n]}_{\mathbf{x}_0}(\mathbf{x}=\mathbf{x}_0) = \sum_{i=1}^{n} \omega_i(\mathbf{x}_0) B^H_i(\langle \mathbf{x}_0, \mathbf{\Theta}_i \rangle) \ ,$$

- Pré-calcul des n B^H_i (complexité $O(\ell \log \ell)$)
- Le reste de l'algorithme s'effectue en $O(\log n \times \#\text{pixels})$



Simulation du LAFBF à H constant

$$B^H_{\alpha,\delta} \leftarrow Y^{[n]}_{\boldsymbol{x}_0}(\boldsymbol{x} = \boldsymbol{x}_0) = \sum_{i=1}^{n} \omega_i(\boldsymbol{x}_0) B_i^H (\langle \boldsymbol{x}_0, \boldsymbol{\Theta}_i \rangle) \ ,$$

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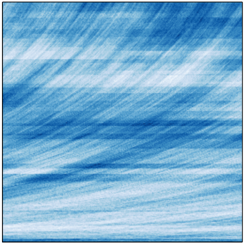
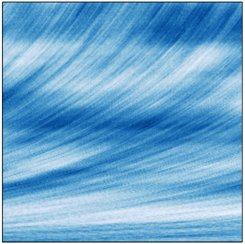


$C_{\boldsymbol{x}_0}(\boldsymbol{\Theta}_i)$

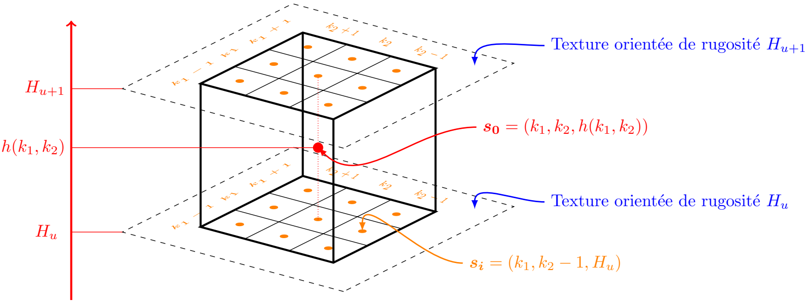


$\widetilde{C}_{\boldsymbol{x}_0}(\boldsymbol{\Theta}_i)$ régularisée

Simulation du LAFBF à h variable (krigeage)

$$\widehat{Z}(\boldsymbol{s_0}) = \sum_{i \in \mathcal{V}(\boldsymbol{s_0})} \lambda_i Z(\boldsymbol{s_i}) = \boldsymbol{\lambda}^{\mathsf{T}} \boldsymbol{Z} \quad (\text{BLUE})$$

$$Z = B^h_{\alpha,\delta},\ (B^{H_u}_{\alpha,\delta})_{1 \leq u \leq U} \rightarrow Z(\boldsymbol{s_i}),\ \boldsymbol{\Sigma}_{ij} = \mathrm{Cov}(Z(\boldsymbol{s_i}), Z(\boldsymbol{s_j})) \rightarrow \boldsymbol{\lambda}$$



Le WAFBF : champs H-sssi déformés

Définition (WAFBF)

Soit X un champ H-sssi et $\mathbf{\Phi} : \mathbb{R}^2 \to \mathbb{R}^2$ une fonction continûment différentiable. Le WAFBF $Z_{\mathbf{\Phi},X}$ est défini comme la déformation du champ X par $\mathbf{\Phi}$:

$$Z_{\mathbf{\Phi},X}(\boldsymbol{x}) = X(\mathbf{\Phi}(\boldsymbol{x})) \ .$$

Références sur les déformations de champs aléatoires stationnaires :

- (Perrin et Senoussi, 1999, 2000)
- (Guyon et Perrin, 2000)

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$$Z_{\mathbf{\Phi},X}(\mathbf{x}) = X(\mathbf{\Phi}(\mathbf{x})) \ .$$

Théorème (Champ tangent d'un WAFBF)

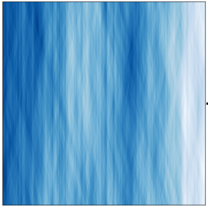
$Z_{\mathbf{\Phi},X}$ admet en tout point $\mathbf{x_0} \in \mathbb{R}^2$ un champ tangent :

$$Y_{\mathbf{x_0}}(\mathbf{x}) = X(\mathbf{D\Phi}(\mathbf{x_0})\ \mathbf{x}) \ , \quad \forall \mathbf{x} \in \mathbb{R}^2 \ ,$$

où $\mathbf{D\Phi}(\mathbf{x_0})$ est la matrice jacobienne de $\mathbf{\Phi}$ au point $\mathbf{x_0}$.

Champ élémentaire déformé

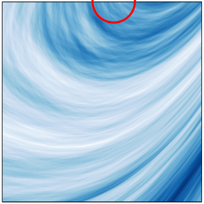
$$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$$



X

$$\xrightarrow[\mathbf{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}]{\boldsymbol{\Phi}(\boldsymbol{x})}$$

WAFBF



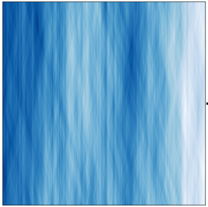
$Z = X \circ \boldsymbol{\Phi}$

1. La directionnalité n'est pas contrôlée

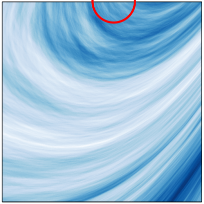
Champ élémentaire déformé

$$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$$

WAFBF



$$\xrightarrow[\mathbf{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}]{\boldsymbol{\Phi}(\boldsymbol{x})}$$

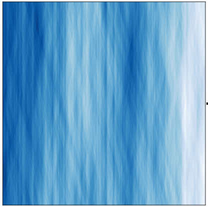


X $\qquad$ $Z = X \circ \boldsymbol{\Phi}$

1. La directionnalité n'est pas contrôlée
2. Quelle transformation $\boldsymbol{\Phi}$ permet de prescrire en tout point les orientations $\alpha(\boldsymbol{x})$?

Champ élémentaire déformé

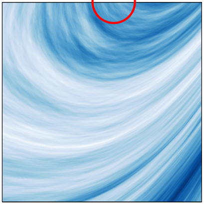
$$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$$



X

$$\xrightarrow[\mathbf{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}]{\boldsymbol{\Phi}(\boldsymbol{x})}$$

WAFBF



$Z = X \circ \boldsymbol{\Phi}$

1. La directionnalité n'est pas contrôlée
2. Quelle transformation $\boldsymbol{\Phi}$ permet de prescrire en tout point les orientations $\alpha(\boldsymbol{x})$?
3. Quelle définition pour l'orientation d'un champ aléatoire ?

Le WAFBF : champs H-sssi déformés

Définition (WAFBF)

Soit X un champ H-sssi et $\mathbf{\Phi} : \mathbb{R}^2 \to \mathbb{R}^2$ une fonction continûment différentiable. Le WAFBF $Z_{\mathbf{\Phi},X}$ est défini comme la déformation du champ X par $\mathbf{\Phi}$:

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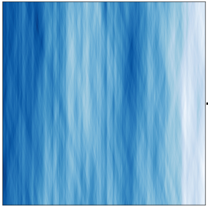
Champ élémentaire déformé

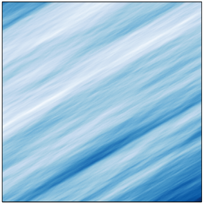
$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$

WAFBF



X

$$\xrightarrow[\mathbf{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}]{\boldsymbol{\Phi}(\boldsymbol{x})}$$

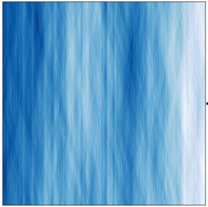


$Z = X \circ \boldsymbol{\Phi}$

$$\alpha(x_1, x_2) = -\frac{\pi}{4}$$

Champ élémentaire déformé

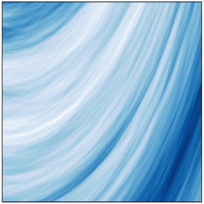
$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$



X

$$\xrightarrow[\mathbf{R}_{\color{red}{-\alpha(\boldsymbol{x})}}\boldsymbol{x}]{\boldsymbol{\Phi}(\boldsymbol{x})}$$

WAFBF

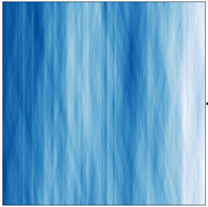


$Z = X \circ \boldsymbol{\Phi}$

$$\color{red}{\alpha(x_1, x_2) = -\frac{\pi}{2} + x_1}$$

Champ élémentaire déformé

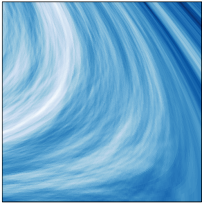
$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$



X

$\xrightarrow[\mathbf{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}]{\boldsymbol{\Phi}(\boldsymbol{x})}$

WAFBF

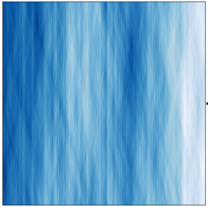


$Z = X \circ \boldsymbol{\Phi}$

$$\alpha(x_1, x_2) = -\frac{\pi}{2} + x_2$$

Champ élémentaire déformé

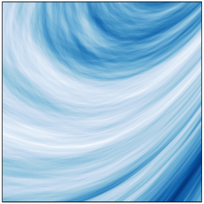
$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$



X

$$\xrightarrow[\mathbf{R}_{-\alpha(\mathbf{x})}\mathbf{x}]{\mathbf{\Phi}(\mathbf{x})}$$

WAFBF



$Z = X \circ \mathbf{\Phi}$

$$\alpha(x_1, x_2) = -\frac{\pi}{2} + x_1^2 - x_2$$

Orientation locale d'une fonction déterministe

Transformée de Riesz et signal monogène (Felsberg, 2001)

L'opérateur de Riesz $\boldsymbol{\mathcal{R}} : f \mapsto (\mathcal{R}_1 f, \mathcal{R}_2 f)$ est défini par :

$$\widehat{\mathcal{R}_1 f}(\boldsymbol{\omega}) = -\mathrm{j}\frac{\omega_1}{\|\boldsymbol{\omega}\|}\widehat{f}(\boldsymbol{\omega}), \quad \widehat{\mathcal{R}_2 f}(\boldsymbol{\omega}) = -\mathrm{j}\frac{\omega_2}{\|\boldsymbol{\omega}\|}\widehat{f}(\boldsymbol{\omega})$$

$\Rightarrow$ Orientation : $\boldsymbol{n}(\boldsymbol{x}) = \dfrac{\boldsymbol{\mathcal{R}} f(\boldsymbol{x})}{\|\boldsymbol{\mathcal{R}} f(\boldsymbol{x})\|}$, $\theta(\boldsymbol{x}) = \arctan\left(\dfrac{\mathcal{R}_2 f(\boldsymbol{x})}{\mathcal{R}_1 f(\boldsymbol{x})}\right)$

$\Rightarrow$ (Plus robuste) minimiser les écarts de direction à $\boldsymbol{\mathcal{R}} f$:

$$\max_{\theta'} \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \left\langle \boldsymbol{n}(\theta'), \boldsymbol{\mathcal{R}} f(\boldsymbol{x}')\right\rangle^2 \mathrm{d}\boldsymbol{x}' = \max_{\theta'} \boldsymbol{n}(\theta')^{\mathsf{T}} \mathbf{J}_f^W(\boldsymbol{x}) \boldsymbol{n}(\theta')$$

$$[\mathbf{J}_f^W(\boldsymbol{x})]_{pq} = \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \mathcal{R}_p f(\boldsymbol{x}') \mathcal{R}_q f(\boldsymbol{x}') \, \mathrm{d}\boldsymbol{x}', \quad p, q \in \{1, 2\}$$

Orientation locale d'une fonction déterministe

Coefficients d'ondelettes monogènes (Unser, Olhede, 2009)

Soit $\psi_{i,\boldsymbol{k}}(\boldsymbol{x}) = 2^i \psi(2^i \boldsymbol{x} - \boldsymbol{k})$ une trame d'ondelettes à partir d'une ondelette réelle isotrope $\widehat{\psi}(\boldsymbol{\xi}) = \varphi(\|\boldsymbol{\xi}\|)$. On considère les coefficients d'ondelettes de $\boldsymbol{\mathcal{R}} f$ dans la trame $\{\psi_{i,\boldsymbol{k}}\}$:

$$c_{i,\boldsymbol{k}}^{(\mathcal{R})}(f) = \begin{pmatrix} c_{i,\boldsymbol{k}}^{(1)}(f) \\ c_{i,\boldsymbol{k}}^{(2)}(f) \end{pmatrix} = \begin{pmatrix} \langle \mathcal{R}_1 f, \psi_{i,\boldsymbol{k}} \rangle \\ \langle \mathcal{R}_2 f, \psi_{i,\boldsymbol{k}} \rangle \end{pmatrix} = \begin{pmatrix} \langle f, \mathcal{R}_1 \psi_{i,\boldsymbol{k}} \rangle \\ \langle f, \mathcal{R}_2 \psi_{i,\boldsymbol{k}} \rangle \end{pmatrix}$$

$$\mathbf{J}_{f,i}^{W}[\boldsymbol{k}] = c_{i,\boldsymbol{k}}^{(\mathcal{R})}(f) c_{i,\boldsymbol{k}}^{(\mathcal{R})}(f)^* = \begin{pmatrix} \frac{|c_{i,\boldsymbol{k}}^{(1)}(f)|^2}{c_{i,\boldsymbol{k}}^{(1)}(f) \cdot c_{i,\boldsymbol{k}}^{(2)}(f)} & c_{i,\boldsymbol{k}}^{(1)}(f) \cdot \overline{c_{i,\boldsymbol{k}}^{(2)}(f)} \\ & |c_{i,\boldsymbol{k}}^{(1)}(f)|^2 \end{pmatrix}$$

Orientation pour un champ H-sssi

Coefficients d'ondelettes monogènes d'un champ H-sssi X

$$c_{i,\boldsymbol{k}}^{(\ell)}(X) = \langle X, \mathcal{R}_\ell \psi_{i,\boldsymbol{k}} \rangle = \int_{\mathbb{R}^2} \widehat{\mathcal{R}_\ell \psi_{i,\boldsymbol{k}}}(\boldsymbol{\xi}) C(\boldsymbol{\xi}) \|\boldsymbol{\xi}\|^{-H-1} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

Théorème (Polisano et coll., 2017)

Soit $\boldsymbol{\Sigma}(c_{i,\boldsymbol{k}}^{(\mathcal{R})}(X)) = \mathbb{E}[c_{i,\boldsymbol{k}}^{(\mathcal{R})}(X) c_{i,\boldsymbol{k}}^{(\mathcal{R})}(X)^*]$, alors :

$$\boldsymbol{\Sigma}(c_{i,\boldsymbol{k}}^{(\mathcal{R})}(X)) = 2^{-2i(H+1)} \left[\int_0^{+\infty} \frac{|\varphi(r)|^2}{r^{2H+1}} \, \mathrm{d}r \right] \mathbf{J}_X \, ,$$

où $\mathbf{J}_X$ est appelée tenseur de structure de X défini par :

$$[\mathbf{J}_X]_{pq} = \int_{\boldsymbol{\Theta} \in \mathbb{S}^1} \Theta_p \Theta_q \; C(\boldsymbol{\Theta})^2 \, \mathrm{d}\boldsymbol{\Theta}, \quad p, q \in \{1, 2\} \, .$$

Calcul d'orientation de champs dérivés d'un EF

- $X = X_{\alpha_0,\delta}$ avec $C(\boldsymbol{\Theta}) = \mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\Theta} - \alpha_0)$ (EF)

$$\vec{\boldsymbol{n}}_X = \begin{pmatrix} \cos \alpha_0 \\ \sin \alpha_0 \end{pmatrix} \stackrel{\text{def}}{=} \boldsymbol{u}(\alpha_0), \quad \chi = \frac{\sin(2\delta)}{2\delta}$$

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- $X = X_{\alpha_0,\delta} + X_{\alpha_1,\delta}$ (somme de 2 EF)

$$\vec{\boldsymbol{n}}_X = \boldsymbol{u}\left(\frac{\alpha_0 + \alpha_1}{2}\right), \quad \chi = \frac{\sin(2\delta)}{2\delta} \cos(\alpha_0 - \alpha_1)$$

Calcul d'orientation de champs dérivés d'un EF

- $X = X_{\alpha_0,\delta}$ avec $C(\mathbf{\Theta}) = \mathbb{1}_{[-\delta,\delta]}(\arg \mathbf{\Theta} - \alpha_0)$ (EF)

$$\vec{\boldsymbol{n}}_X = \begin{pmatrix} \cos \alpha_0 \\ \sin \alpha_0 \end{pmatrix} \overset{\text{def}}{=} \boldsymbol{u}(\alpha_0), \quad \chi = \frac{\sin(2\delta)}{2\delta}$$

- $X = X_{\alpha_0,\delta} + X_{\alpha_1,\delta}$ (somme de 2 EF)

$$\vec{\boldsymbol{n}}_X = \boldsymbol{u}\left(\frac{\alpha_0 + \alpha_1}{2}\right), \quad \chi = \frac{\sin(2\delta)}{2\delta} \cos(\alpha_0 - \alpha_1)$$

- $X_{\mathbf{L}}(\boldsymbol{x}) = X_{\alpha_0,\delta}(\mathbf{L}\boldsymbol{x})$ (déformation linéaire d'un EF)

$$\vec{\boldsymbol{n}}_{X_{\mathbf{L}}} = \frac{\mathbf{L}^{\mathsf{T}}\boldsymbol{u}(\alpha_0)}{\|\mathbf{L}^{\mathsf{T}}\boldsymbol{u}(\alpha_0)\|}$$

Orientation d'un champ gaussien localisable

Orientation locale du WAFBF où $X = X_{\alpha_0,\delta}$ est un EF

Le champ tangent de $Z_{\mathbf{\Phi},X}(\boldsymbol{x}) = X_{\alpha_0,\delta}(\mathbf{\Phi}(\boldsymbol{x}))$ en $\boldsymbol{x}_0$ est

$$Y_{\boldsymbol{x}_0}(\boldsymbol{x}) = X_{\alpha_0,\delta}(\mathbf{D\Phi}(\boldsymbol{x}_0)\ \boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^2$$

dont l'orientation est $\vec{\boldsymbol{n}}_{Y_{\boldsymbol{x}_0}} = \frac{\mathbf{L}^{\mathsf{T}}\boldsymbol{u}(\alpha_0)}{\|\mathbf{L}^{\mathsf{T}}\boldsymbol{u}(\alpha_0)\|}$ avec $\mathbf{L} = \mathbf{D\Phi}(\boldsymbol{x}_0)$, d'où

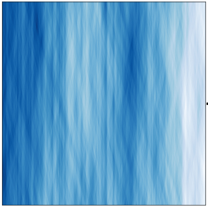
$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x}_0) \equiv \vec{\boldsymbol{n}}_{Y_{\boldsymbol{x}_0}} = \frac{\mathbf{D\Phi}(\boldsymbol{x}_0)^{\mathsf{T}}\boldsymbol{u}(\alpha_0)}{\|\mathbf{D\Phi}(\boldsymbol{x}_0)^{\mathsf{T}}\boldsymbol{u}(\alpha_0)\|}$$

Orientation d'un champ gaussien localisable

Exemple (Rotation locale du WAFBF où $X = X_{0,\delta}$)

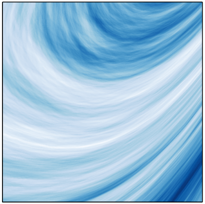
L'orientation locale de $Z_{\mathbf{\Phi},X}(x) = X_{0,\delta}(\mathbf{\Phi}(x))$ en $\boldsymbol{x_0}$ avec $\mathbf{\Phi}(\boldsymbol{x}) = \mathbf{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}$ est donnée par $\vec{\boldsymbol{n}}_Z(\boldsymbol{x_0}) = \frac{\mathbf{D\Phi}(\boldsymbol{x_0})^{\mathsf{T}}\boldsymbol{e_1}}{\|\mathbf{D\Phi}(\boldsymbol{x_0})^{\mathsf{T}}\boldsymbol{e_1}\|}$ soit :

$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x_0}) = \frac{\boldsymbol{u}(\alpha(\boldsymbol{x_0})) + \langle \boldsymbol{u}(\alpha(\boldsymbol{x_0}))^{\perp}, \boldsymbol{x_0}\rangle \nabla\alpha(\boldsymbol{x_0})}{\|\boldsymbol{u}(\alpha(\boldsymbol{x_0})) + \langle \boldsymbol{u}(\alpha(\boldsymbol{x_0}))^{\perp}, \boldsymbol{x_0}\rangle \nabla\alpha(\boldsymbol{x_0})\|}$$



$X_{0,\delta}$

$$\xrightarrow[\mathbf{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}]{\mathbf{\Phi}(\boldsymbol{x})}$$



$Z_{\mathbf{\Phi},X} = X_{0,\delta} \circ \mathbf{\Phi}$

Orientations prescrites pour un WAFBF

Proposition (Contrôle d'orientation par fonctions harmoniques)

Soit $Z_{\mathbf{\Phi}_\alpha,X}(\boldsymbol{x})$ le champ $X = X_{0,\delta}$ d'orientation $\boldsymbol{e}_1 = (1,0)^{\mathsf{T}}$ déformé par une transformation conforme $\mathbf{\Phi}_\alpha$ définie par :

1. $\alpha : \mathbb{R}^2 \to \mathbb{R}$ une fonction harmonique,
2. λ sa fonction conjuguée harmonique telle que $\mathbf{\Psi}_\alpha = \begin{pmatrix} \lambda \\ -\alpha \end{pmatrix}$ est holomorphe,
3. $\mathbf{\Phi}_\alpha$ une primitive complexe de $\exp(\mathbf{\Psi}_\alpha)$.

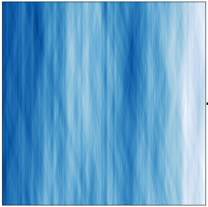
L'orientation locale (à δ^2 près) de $Z_{\mathbf{\Phi}_\alpha,X}$ en $\boldsymbol{x_0}$ est

$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x_0}) = \begin{pmatrix} \cos\alpha(\boldsymbol{x_0}) \\ \sin\alpha(\boldsymbol{x_0}) \end{pmatrix} = \boldsymbol{u}(\alpha(\boldsymbol{x}_0))$$

Orientations prescrites pour un WAFBF

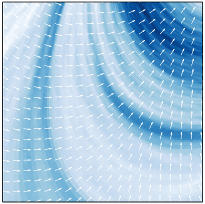
$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$

WAFBF $(a, b) = (2, -1)$



$\boldsymbol{\Phi}_\alpha$

$\alpha(x_1, x_2) = ax_1 + bx_2 + c$



$X_{0,\delta}$

$Z = X_{0,\delta} \circ \boldsymbol{\Phi}_\alpha$

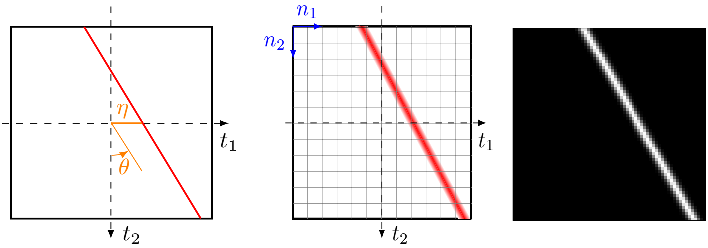
$$\boldsymbol{\Phi}_\alpha(x_1, x_2) = \frac{\exp(ax_2 - bx_1)}{a^2 + b^2} \begin{pmatrix} a\sin(ax_1 + bx_2 + c) - b\cos(ax_1 + bx_2 + c) \\ a\cos(ax_1 + bx_2 + c) + b\sin(ax_1 + bx_2 + c) \end{pmatrix}$$

$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x}) = \frac{\mathbf{D}\boldsymbol{\Phi}(\boldsymbol{x})^\mathsf{T}\boldsymbol{e}_1}{\|\mathbf{D}\boldsymbol{\Phi}(\boldsymbol{x})^\mathsf{T}\boldsymbol{e}_1\|} = \boldsymbol{u}(\alpha(\boldsymbol{x}))$$

Modélisation des droites

$$x^\sharp : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^{K} \alpha_k \delta\big(\cos\theta_k (t_1 - \eta_k) + \sin\theta_k \, t_2\big)$$

$$\mathbf{b}^\sharp[n_1, n_2] = (x^\sharp * \phi)(n_1, n_2)$$

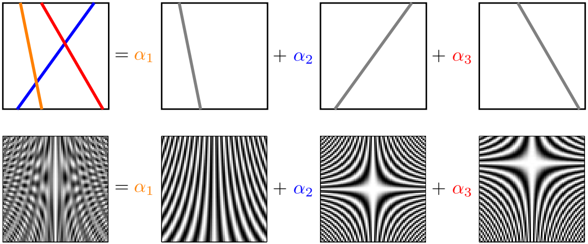


$x^\sharp$ $\qquad$ $x^\sharp * \phi$ $\qquad$ $\mathbf{b}^\sharp$

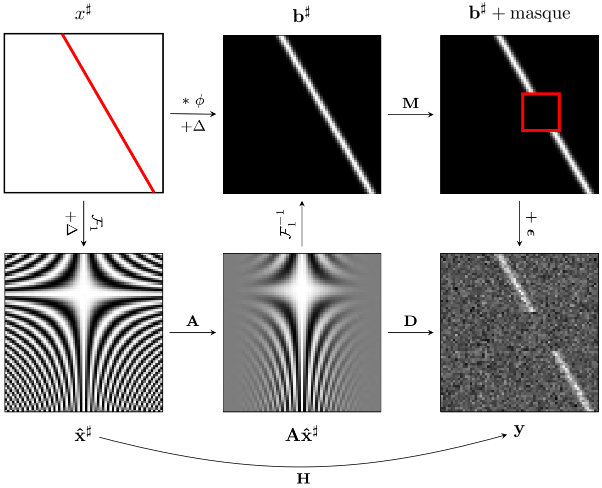
Modélisation des droites

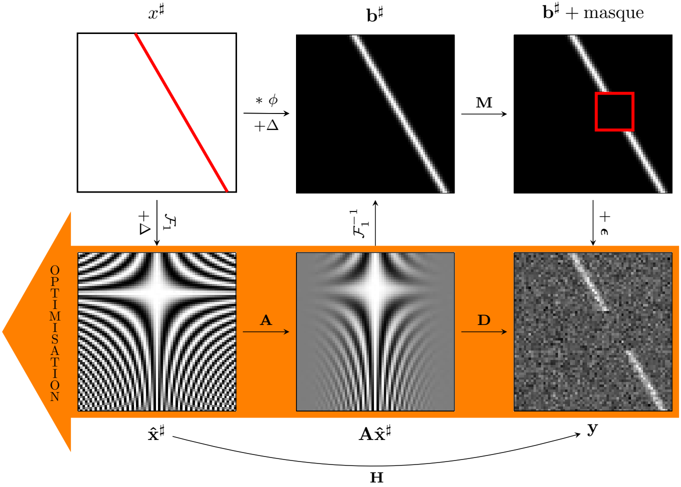
$$\widehat{\mathbf{x}}^\sharp[m, n_2] = (\mathcal{F}_1 x^\sharp)[m, n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right) m} \qquad c_k = \frac{\alpha_k}{\cos\theta_k} \geq$$

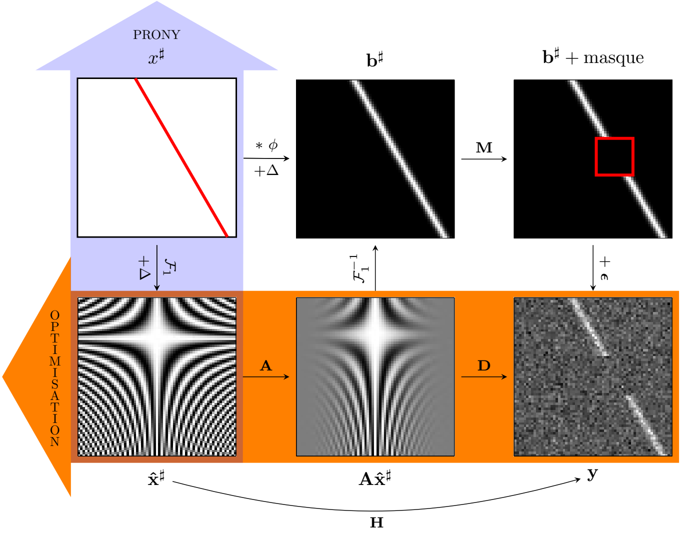
$$\widehat{\mathbf{b}}^\sharp[m, :] = (\widehat{g}[m]\widehat{\mathbf{x}}^\sharp[m, :]) * \boldsymbol{h} \rightarrow \mathbf{A}\widehat{\mathbf{x}}^\sharp = \widehat{\mathbf{b}}^\sharp$$



Étapes de reconstruction




$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$* \ \phi$
$+\Delta$
$\mathbf{M}$
$+\Delta$
$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
$+\epsilon$
OPTIMISATION
$\mathbf{A}$
$\mathbf{D}$
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
$\mathbf{H}$


PRONY
$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$*\ \phi$
$+\Delta$
$\mathbf{M}$
$+\Delta$
$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
$+\epsilon$
OPTIMISATION
$\mathbf{A}$
$\mathbf{D}$
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
$\mathbf{H}$

Décomposition atomique des lignes et colonnes

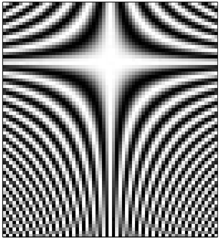
$$\widehat{\mathbf{x}}^\sharp[m, n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right)m}$$

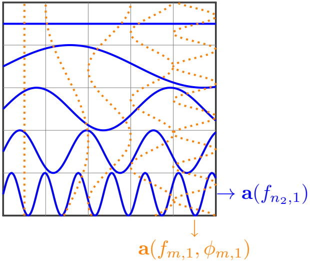
1. $l^\sharp_{n_2} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k}, 0)$ (lignes de $\widehat{\mathbf{x}}$, sans phase)
2. $t^\sharp_{m} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^\mathsf{T}$ (colonnes de $\widehat{\mathbf{x}}$, avec phase)

Décomposition atomique d'une droite ($K = 1$)

$$\widehat{\mathbf{x}}^\sharp[m, n_2] = c_1 \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_1}{W} n_2 + \frac{\eta_1}{W}\right)m}$$

1. $\boldsymbol{l}^\sharp_{n_2} = c_1 \boldsymbol{a}(f_{n_2,1}, 0)$ (un atome sans phase)
2. $\boldsymbol{t}^\sharp_m = c_1 \boldsymbol{a}(f_{m,1}, \phi_{m,1})^\mathsf{T}$ (un atome avec phase)





Normes atomiques

$$\widehat{\mathbf{x}}^\sharp[m,n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W}n_2+\frac{\eta_k}{W}\right)m}, \quad c^\star = \sum_{k=1}^{K} c_k$$

1. $l_{n_2}^\sharp = \sum_{k=1}^K c_k \boldsymbol{a}(f_{n_2,k}, 0)$ (lignes de $\widehat{\mathbf{x}}$, sans phase)
2. $t_m^\sharp = \sum_{k=1}^K c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^\mathsf{T}$ (colonnes de $\widehat{\mathbf{x}}$, avec phase)

Norme atomique :

$$\|\boldsymbol{z}\|_{\mathcal{A}} = \inf_{c'_k, f'_k, \phi'_k} \left\{ \sum_k c'_k : \boldsymbol{z} = \sum_k c'_k \boldsymbol{a}(f'_k, \phi'_k) \right\}$$

Normes atomiques

1. $\boldsymbol{l}^{\sharp}_{n_2} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k}, 0)$

$\hookrightarrow \mathbf{T}_{M+1}(\boldsymbol{l}^{\sharp}_{n_2}) \succcurlyeq 0$ + de rang K (Carathéodory, 1907)

$\hookrightarrow \|\boldsymbol{l}^{\sharp}_{n_2}\|_{\mathcal{A}} = \sum_{k=1}^{K} c_k = \hat{\mathbf{x}}^{\sharp}[0, n_2]$

2. $\boldsymbol{t}^{\sharp}_{m} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^{\mathsf{T}}$ (Tang et coll., 2013)

$$\|\boldsymbol{t}^{\sharp}_{m}\|_{\mathcal{A}} = \inf_{\boldsymbol{q}\in\mathbb{C}^N, t\in\mathbb{R}} \left\{ \frac{1}{2}\mathrm{Tr}(\mathbf{T}_N(\boldsymbol{q})) + \frac{1}{2}t : \begin{pmatrix} \mathbf{T}_N(\boldsymbol{q}) & \boldsymbol{t}^{\sharp}_{m} \\ \boldsymbol{t}^{\sharp\,*}_{m} & t \end{pmatrix} \succcurlyeq 0 \right\} .$$

Normes atomiques

1. $\boldsymbol{l}^\sharp_{n_2} = \sum_{k=1}^K c_k \boldsymbol{a}(f_{n_2,k}, 0)$

$\hookrightarrow$ $\mathbf{T}_{M+1}(\boldsymbol{l}^\sharp_{n_2}) \succcurlyeq 0$ + de rang K (Carathéodory, 1907)

$\hookrightarrow$ $\|\boldsymbol{l}^\sharp_{n_2}\|_{\mathcal{A}} = \sum_{k=1}^K c_k = \hat{\mathbf{x}}^\sharp[0, n_2] = c^\star$

2. $\boldsymbol{t}^\sharp_m = \sum_{k=1}^K c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^\mathsf{T}$ (Polisano et coll., 2016)

$$\|\boldsymbol{t}^\sharp_m\|_{\mathcal{A}} = \min_{\boldsymbol{q}\in\mathbb{C}^N} \left\{ q_0 : \underbrace{\begin{pmatrix} \mathbf{T}_N(\boldsymbol{q}) & \boldsymbol{t}^\sharp_m \\ \boldsymbol{t}^{\sharp\,*}_m & q_0 \end{pmatrix}}_{\mathbf{T}'_N(\boldsymbol{t}^\sharp_m, \boldsymbol{q})} \succcurlyeq 0 \right\} \equiv \mathrm{SDP}(\boldsymbol{t}^\sharp_m)\,,$$

$\hookrightarrow$ $\|\boldsymbol{t}^\sharp_m\|_{\mathcal{A}} = \mathrm{SDP}(\boldsymbol{t}^\sharp_m) = \boldsymbol{q}_m[0] \leqslant c^\star$

Normes atomiques

$$\hat{\mathbf{x}}^\sharp[m, n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W}\,n_2 + \frac{\eta_k}{W}\right)m}, \quad c^\star = \sum_{k=1}^{K} c_k$$

1. $\boldsymbol{l}_{n_2}^\sharp = \sum_{k=1}^K c_k \boldsymbol{a}(f_{n_2,k}, 0)$ (lignes de $\hat{\mathbf{x}}$, sans phase)
2. $\boldsymbol{t}_m^\sharp = \sum_{k=1}^K c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^\mathsf{T}$ (colonnes de $\hat{\mathbf{x}}$, avec phase)

Caractérisation (convexe) des K droites par la norme atomique

1. $\|\boldsymbol{l}_{n_2}^\sharp\|_{\mathcal{A}} = c^\star = \hat{\mathbf{x}}^\sharp[0, n_2]$ et $\mathbf{T}_{M+1}(\boldsymbol{l}_{n_2}^\sharp) \succcurlyeq 0$
2. $\|\boldsymbol{t}_m^\sharp\|_{\mathcal{A}} = \mathrm{SDP}(\boldsymbol{t}_m^\sharp) = \boldsymbol{q}_m[0] \leqslant c^\star$, $\mathbf{T}'_{H_S}(\boldsymbol{t}_m^\sharp, \boldsymbol{q}_m) \succcurlyeq 0$

Problème d'optimisation convexe

Proposition (Minimisation convexe)

$$\tilde{\mathbf{x}} \in \underset{\hat{\mathbf{x}},\mathbf{q} \in \mathcal{X} \times \mathcal{Q}}{\arg\min} \frac{1}{2} \|\mathbf{A}\hat{\mathbf{x}} - \hat{\mathbf{y}}\|^2 ,$$

$$\text{sous contraintes} \quad \begin{cases} \forall n_2 = 0, \ldots, H_S - 1, \ \forall m = 0, \ldots, M , \\ \hat{\mathbf{x}}[0, n_2] = \hat{\mathbf{x}}[0, 0] \leqslant c , \\ \mathbf{q}[m, 0] \leqslant c , \\ \mathbf{T}'_{H_S}(\hat{\mathbf{x}}[m, :], \mathbf{q}[m, :]) \succcurlyeq 0 , \\ \mathbf{T}_{M+1}(\hat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 . \end{cases}$$

Problème d'optimisation convexe

Proposition (Minimisation convexe)

$$\tilde{\mathbf{x}} \in \underset{\hat{\mathbf{x}},\mathbf{q}\in\mathcal{X}\times\mathcal{Q}}{\arg\min}\ \frac{1}{2}\|\mathbf{A}\hat{\mathbf{x}} - \hat{\mathbf{y}}\|^2 ,$$

$$\text{sous contraintes} \quad \begin{cases} \forall n_2 = 0, \ldots, H_S - 1,\ \forall m = 0, \ldots, M , \\ \hat{\mathbf{x}}[0, n_2] = \hat{\mathbf{x}}[0, 0] \leqslant c , \\ \mathbf{q}[m, 0] \leqslant c , \\ \mathbf{T}'_{H_S}(\hat{\mathbf{x}}[m, :], \mathbf{q}[m, :]) \succcurlyeq 0 , \\ \mathbf{T}_{M+1}(\hat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 . \end{cases}$$

(Chambolle et Pock, 2010)

$$\tilde{\mathbf{X}} = \underset{\mathbf{X}\in\mathcal{H}}{\arg\min} \left\{ F(\mathbf{X}) + G(\mathbf{X}) + \sum_{i=0}^{Q-1} H_i(\mathrm{L}_i(\mathbf{X})) \right\}$$

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$$\text{sous contraintes} \quad \begin{cases} \forall n_2 = 0, \dots, H_S - 1, \ \forall m = 0, \dots, M , \\ \hat{\mathbf{x}}[0, n_2] = \hat{\mathbf{x}}[0, 0] \leqslant c , \\ \mathbf{q}[m, 0] \leqslant c , \\ \mathbf{T}'_{H_S}(\hat{\mathbf{x}}[m, :], \mathbf{q}[m, :]) \succcurlyeq 0 , \\ \mathbf{T}_{M+1}(\hat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 . \end{cases}$$

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