# Riesz-based orientation of localizable Gaussian fields

Kévin Polisano

*joint work with M. Clausel, L. Condat and V. Perrier*

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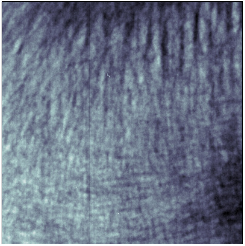
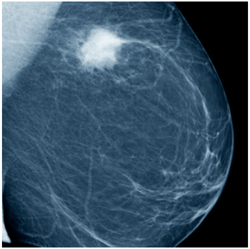
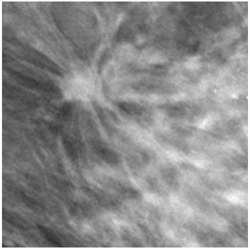
LABORATOIRE JEAN KUNTZMANN
MATHÉMATIQUES APPLIQUÉES - INFORMATIQUE

# Motivation: modelling/analysis anisotropic textures



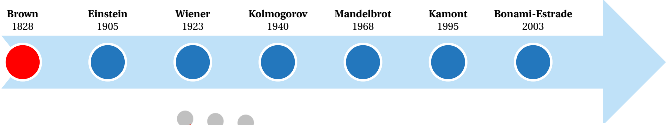




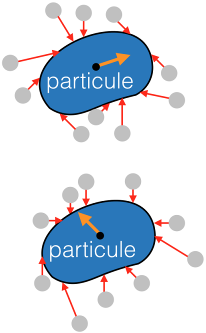

# Outline

1. Intro: from Brownian motion to anisotropic random fields
2. Two classes of Gaussian fields with prescribed orientation:
   - Generalized Anisotropic Fractional Brownian Fields (GAFBF)
   - Warped Anisotropic Fractional Brownian Field (WAFBF)
3. Definition of the notion of orientation for random fields:
   - H-self-similar Gaussian fields with stationary increments (H-sssi)
   - Generalization to the class of localizable Gaussian fields
4. Conclusion and perspectives

# Introduction: from Brownian motion to anisotropic random fields

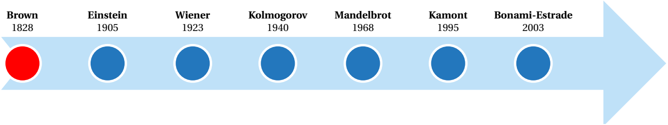
# From Brownian to random anisotropic fields

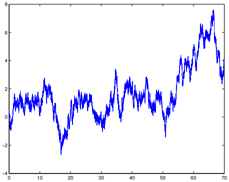


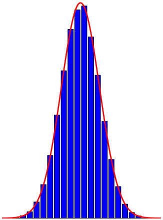




- independants displacements
- Gaussian distribution
- irregular trajectories

# From Brownian to random anisotropic fields



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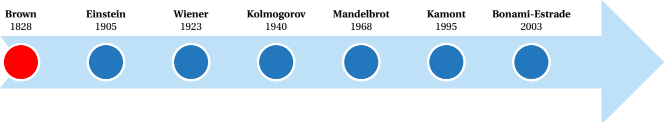


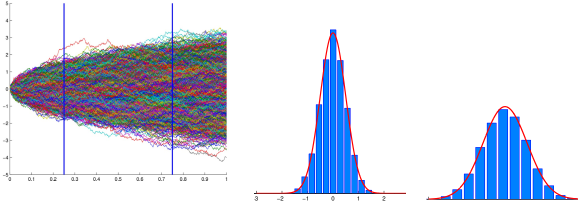

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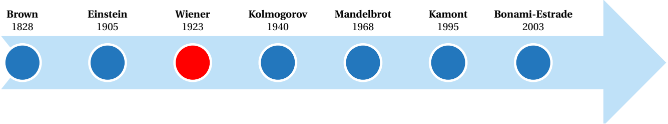
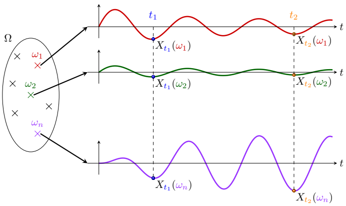



## Brownian motion

- $(B_t)_t$ has independants increments, $B_0 = 0$ a.s.
- $B_{t_i} - B_{t_j} \sim \mathcal{N}(0, t_i - t_j)$
- $(B_t)_t$ has continuous sample paths a.s.

$$X \; : \; \begin{array}{ccc} T \times \Omega & \longrightarrow & E \\ (t,\omega) & \longmapsto & X(t,\omega) \end{array}$$



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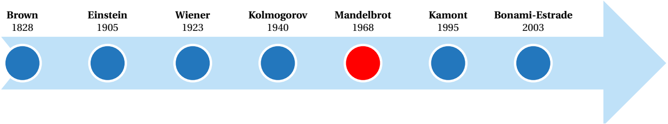
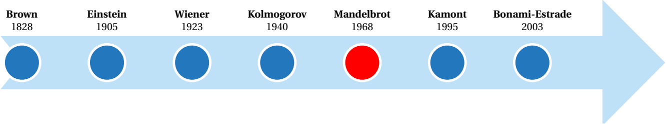



## Self-similarity

$\{X(t)\}_{t\in T}$ is self-similar of order $H$ if $\forall \lambda \in \mathbb{R}$

$$\{X(\lambda t)\}_{t\in T} \overset{(fdd)}{=} \lambda^H \{X(t)\}_{t\in T}$$

# From Brownian to random anisotropic fields

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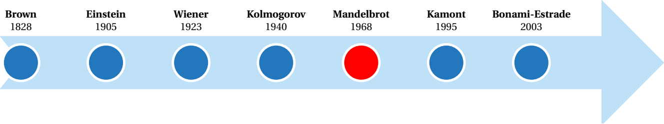


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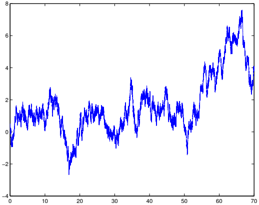

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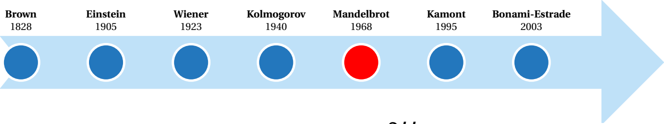
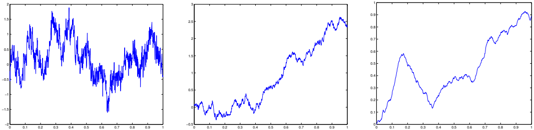


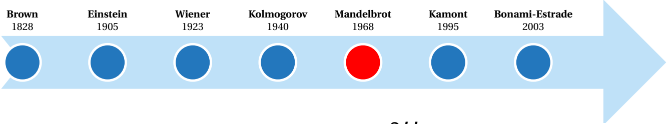

- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ ~~indpt. increments~~

fractional Brownian motion $B^H$ (FBM)



$H = 0.2$ $H = 0.5$ $H = 0.8$

# From Brownian to random anisotropic fields

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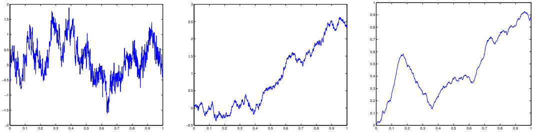

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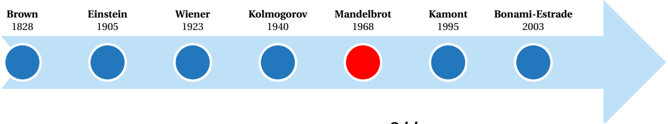
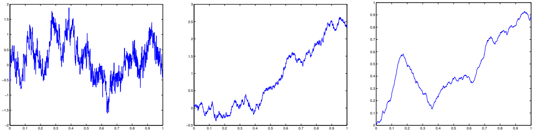



- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ stat. increments
- $\mathrm{R}(t,s) = \mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$

fractional Brownian motion $B^H$ (FBM)



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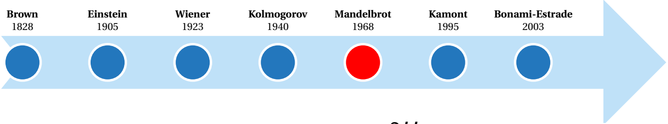


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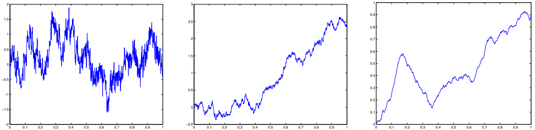
- $\mathrm{R}(t,s) = \mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$
- $B^H(t) = \frac{1}{c_H} \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{j}t\xi} - 1}{|\xi|^{H+1/2}} \widehat{\mathrm{W}}(\xi) \Rightarrow$ harmonizable formula

fractional Brownian motion $B^H$ (FBM)



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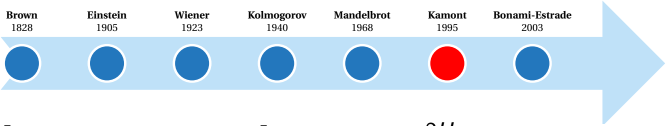
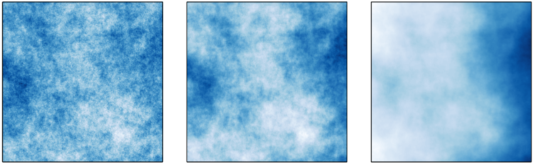



- $\mathbb{E}\left[(B^H(\boldsymbol{x}) - B^H(\boldsymbol{y}))^2\right] = \|\boldsymbol{x}-\boldsymbol{y}\|^{2H}$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$
- $\mathrm{R}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\left(\|\boldsymbol{x}\|^{2H} + \|\boldsymbol{y}\|^{2H} - \|\boldsymbol{x}-\boldsymbol{y}\|^{2H}\right)$
- $B^H(\boldsymbol{x}) = \frac{1}{C_H}\int_{\mathbb{R}^2} \frac{\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1}{\|\boldsymbol{\xi}\|^{H+1}} \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi})$

fractional Brownian field $B^H$ (FBF)



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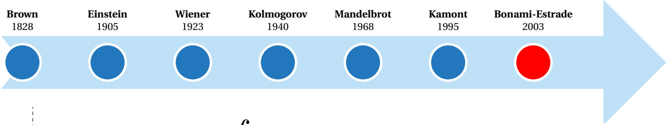
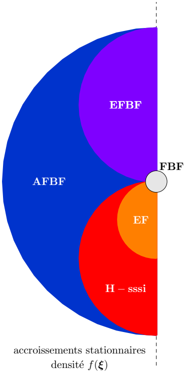
# Model of Bonami-Estrade




$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{\xi}) \, \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi})$$

- $f^{1/2}(\boldsymbol{\xi}) = \dfrac{C}{\|\boldsymbol{\xi}\|^{H+1}}$ (FBF)
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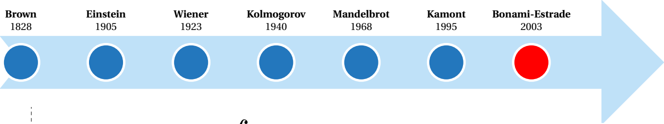


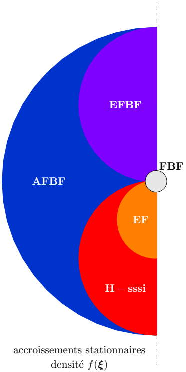


$$X(x) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{\xi}) \, \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi})$$

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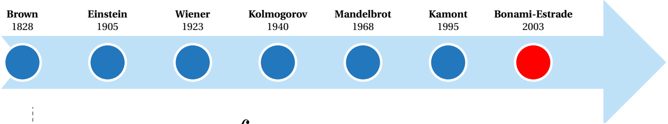


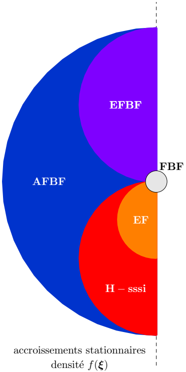


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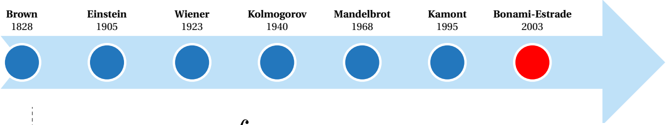


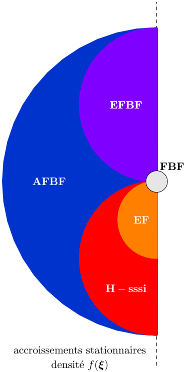


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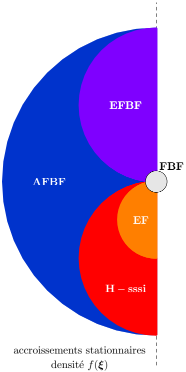
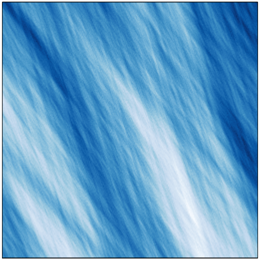
- $f^{1/2}(\xi) = \dfrac{C(\xi)}{\|\xi\|^{h(\xi)+1}}$ (AFBF)

# A special case of H-sssi: the elementary field

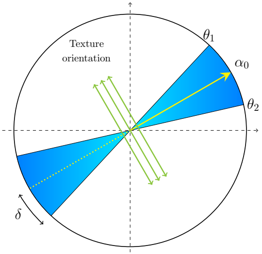
$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(e^{j\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1\right) \frac{\mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha_0)}{\|\boldsymbol{\xi}\|^{H+1}} \widehat{W}(d\boldsymbol{\xi})$$




*Elementary field* (EF) $[H = 0.5,\ \alpha_0 = \pi/6]$



$\delta = 3.10^{-1}$

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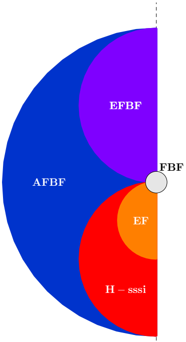


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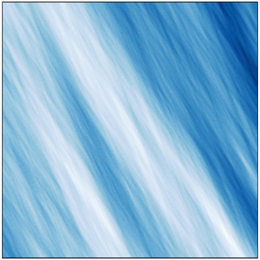

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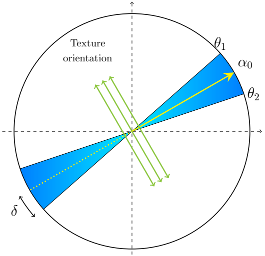


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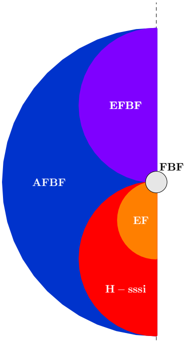


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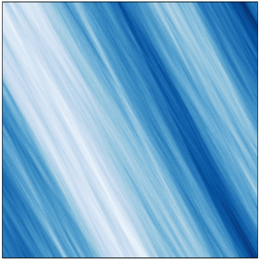

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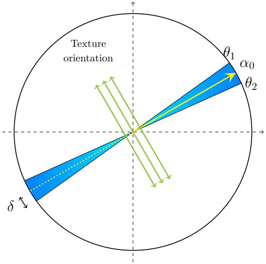


$\delta = 3.10^{-1}$

# State of the art: anisotropic Gaussian fields

- Fractional Brownian sheet (FBS) (Kamont, 1995), (Léger and Pontier, 1999), (Ayache et al., 2002)
- H-sssi fields (Benassi et coll., 1997)
- Model of Bonami and Estrade (Bonami and Estrade, 2003)
- Operator scaling Gaussian random fields (OSGRF) (Schertzer and Lovejoy, 1985), (Biermé et. al, 2007)
- Model of Xue, Xiao, Li (Xue and Xiao, 2011), (Li and Xiao, 2011)
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$\Rightarrow$ no class of fields with controlled local anisotropy

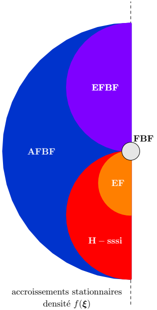
$\Rightarrow$ contribution : two new classes of this type
the (GAFBF) and the (WAFBF)

# Two models: localized and warped H-sssi fields

# From H-sssi fields to GAFBF

$$X(\mathbf{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \mathbf{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{\xi}) \, \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi})$$

If $X$ is $H$–self-similar, that is $X(\lambda \mathbf{x}) = \lambda^H X(\mathbf{x})$, one has:




**H-sssi**

$$f^{1/2}(\boldsymbol{\xi}) = \frac{C(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{H+1}}$$

**EF**

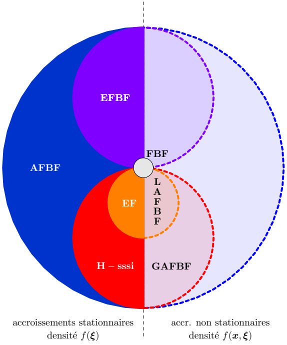
$$C(\boldsymbol{\xi}) = \mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha_0)$$

with homogeneous anisotropic function $\boldsymbol{\xi} \mapsto C(\boldsymbol{\xi})$

# Model with prescribed orientations and regularities

New model: a localized and multifractional version of H-sssi fields

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1\right) f^{1/2}(\boldsymbol{x}, \boldsymbol{\xi}) \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi})$$




**GAFBF**

$$f^{1/2}(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{C(\boldsymbol{x}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x})+1}}$$

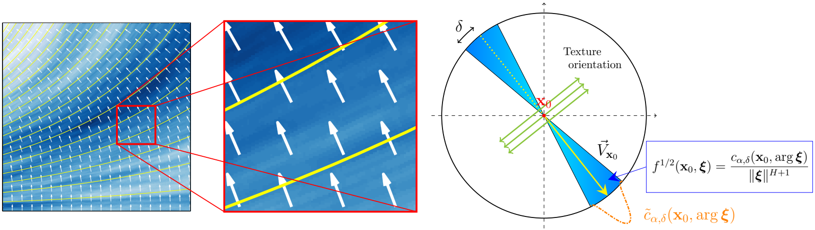
**LAFBF**

$$C(\boldsymbol{x}, \boldsymbol{\xi}) = \mathbb{1}_{[-\delta(\boldsymbol{x}), \delta(\boldsymbol{x})]}(\arg \boldsymbol{\xi} - \alpha(\boldsymbol{x}))$$

# Model with prescribed local orientation

$$B^H_{\alpha,\delta}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(e^{j\langle \boldsymbol{x},\boldsymbol{\xi}\rangle} - 1\right) \frac{\mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha(\boldsymbol{x}))}{\|\boldsymbol{\xi}\|^{H+1}} \widehat{W}(d\boldsymbol{\xi})$$

*localized elementary field* (LAFBF) $[H = 0.8,\ \alpha(x_1, x_2) = -\pi/2 + x_1]$




(a) (b) (c)

# The tangent field: a tool for analysis and synthesis

1. A tool for analysis (Lévy-Vehel, 1995), (Falconer, 2002) :

$$\left\{ \lim_{\rho \to 0} \frac{X(\boldsymbol{x}_0 + \rho \boldsymbol{x}) - X(\boldsymbol{x}_0)}{\rho^{h(\boldsymbol{x}_0)}} \right\}_{\boldsymbol{x} \in \mathbb{R}^2} \overset{d}{=} \{ Y_{\boldsymbol{x}_0}(\boldsymbol{x}) \}_{\boldsymbol{x} \in \mathbb{R}^2}$$

   Roughly speaking $Y_{\boldsymbol{x}_0}$ is the "local form" of $X$ at point $\boldsymbol{x}_0$.

2. A tool for synthesis (Lévy-Véhel, 1995), (Benassi, 1997) :

$$X(\boldsymbol{x}_0) \leftarrow Y_{\boldsymbol{x}_0}(\boldsymbol{x} = \boldsymbol{x}_0)$$

$\Rightarrow$ If $Y$ is "localizable", all local anisotropy characteristics are defined and herited from its tangent field.

# Assumptions on the GAFBF

## Assumptions $(\mathcal{H})$

- $h$ is $\beta$–Hölder, such that $a = \inf_{\boldsymbol{x}\in\mathbb{R}^2} h(\boldsymbol{x}) > 0$, $b = \sup_{\boldsymbol{x}\in\mathbb{R}^2} h(\boldsymbol{x})$ and $b < \beta \leq 1$.
- $(\boldsymbol{x},\boldsymbol{\xi}) \mapsto C(\boldsymbol{x},\boldsymbol{\xi})$ is bounded $C(\boldsymbol{x},\boldsymbol{\xi}) \leqslant M, \forall(\boldsymbol{x},\boldsymbol{\xi})$.
- $\boldsymbol{\xi} \mapsto C(\boldsymbol{x},\boldsymbol{\xi})$ is even $C(\boldsymbol{x},-\boldsymbol{\xi}) = C(\boldsymbol{x},\boldsymbol{\xi})$.
- $\boldsymbol{\xi} \mapsto C(\boldsymbol{x},\boldsymbol{\xi})$ homogeneous $C(\boldsymbol{x},\rho\boldsymbol{\xi}) = C(\boldsymbol{x},\boldsymbol{\xi}), \forall\rho$.
- $\boldsymbol{x} \mapsto C(\boldsymbol{x},\boldsymbol{\xi})$ is continuous and $\exists\eta$, $\beta \leq \eta \leq 1$, $\forall\boldsymbol{x}$

$$\sup_{\boldsymbol{z}\in B(0,1)} \|\boldsymbol{z}\|^{-2\eta} \int_{\mathbb{S}^1} \left[C(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{\Theta}) - C(\boldsymbol{x},\boldsymbol{\Theta})\right]^2 \mathrm{d}\boldsymbol{\Theta} \leq A_{\boldsymbol{x}} < \infty$$

# Tangent field of the GAFBF

Let $X$ be the GAFBF defined by

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1\right) \frac{C(\boldsymbol{x}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x})+1}} \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi})$$

**Theorem (P. et al., 2017)**

If $X$ satisfies the assumptions $(\mathcal{H})$, then $X$ admits at every point $\boldsymbol{x}_0 \in \mathbb{R}^2$ a tangent field $Y_{\boldsymbol{x}_0}$ given by:

$$\begin{aligned} Y_{\boldsymbol{x}_0}(\boldsymbol{x}) &= \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{x}_0, \boldsymbol{\xi}) \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi}) \ , \\ &= \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C(\boldsymbol{x}_0, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x}_0)+1}} \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi}) \ . \end{aligned}$$

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$$\text{H-sssi field} = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C_{\boldsymbol{x}_0}(\xi)}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x}_0)+1}} \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi}) \ .$$

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**Multifractional Brownian field $B^h$ (MBF)** (Peltier, Vehel, 1995)

- Analysis : the MBF behaves locally as a FBF

$$\left\{ \lim_{\rho \to 0} \frac{B^h(\boldsymbol{x}_0 + \rho \boldsymbol{x}) - B^h(\boldsymbol{x}_0)}{\rho^{h(\boldsymbol{x}_0)}} \right\}_{\boldsymbol{x} \in \mathbb{R}^2} \overset{d}{=} \left\{ B^{h(\boldsymbol{x}_0)}(\boldsymbol{x}) \right\}_{\boldsymbol{x} \in \mathbb{R}^2}$$

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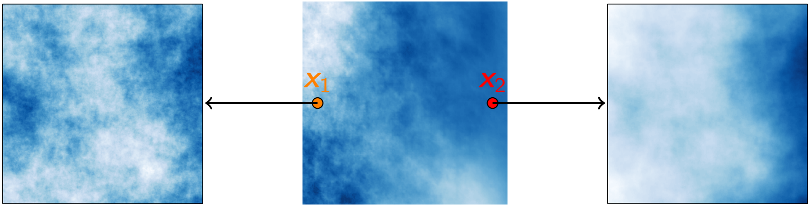
$B^H$, $H \equiv h(\boldsymbol{x}_1)$ MBM $B^{h(\boldsymbol{x})}$ $B^H$, $H \equiv h(\boldsymbol{x}_2)$



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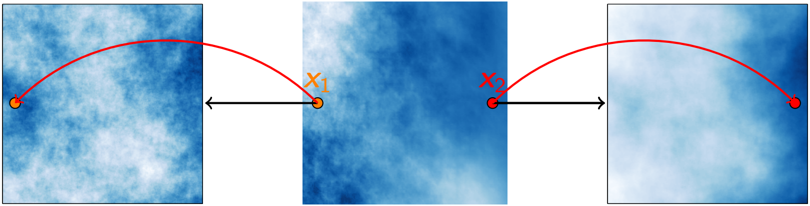
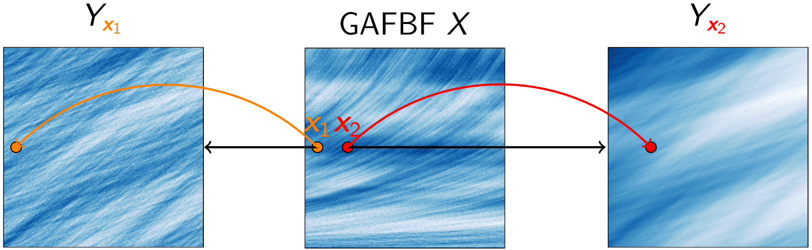
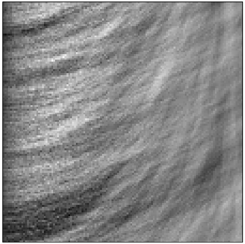


# Synthesis of the GAFBF by its tangent fields

$$X(\boldsymbol{x}_0) \leftarrow Y_{\boldsymbol{x}_0}(\boldsymbol{x} = \boldsymbol{x}_0) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C_{\boldsymbol{x}_0}(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x}_0)+1}} \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi})$$



# Simulation of the LAFBF



- Linear variation of the orientations $\alpha(\boldsymbol{x})$ along $(Ox)$
- Linear variation of the directionality $\delta(\boldsymbol{x})$ along $(Ox)$
- Linear variation of the regularity $h(\boldsymbol{x})$ along $(Ox)$

# The WAFBF: warped H-sssi fields

## Definition (WAFBF)

Let $X$ be a H-sssi field and $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function. The *Warped Anisotropic Fractional Brownian Field* (WAFBF) $Z_{\Phi,X}$ is defined as the deformation of the elementary field $X$ by the application $\Phi$:

$$Z_{\Phi,X}(\boldsymbol{x}) = X(\Phi(\boldsymbol{x})) \ .$$

References about deformations of stationary random fields:

- (Perrin and Senoussi, 1999, 2000)
- (Guyon and Perrin, 2000)

# The WAFBF: warped H-sssi fields

## Definition (WAFBF)

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$$Z_{\Phi,X}(\boldsymbol{x}) = X(\Phi(\boldsymbol{x})) \ .$$

## Theorem (Tangent field of the WAFBF)

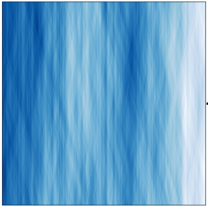
$Z_{\Phi,X}$ admits at every point $\boldsymbol{x}_0 \in \mathbb{R}^2$ the tangent field:

$$Y_{\boldsymbol{x}_0}(\boldsymbol{x}) = X(\mathrm{D}\Phi(\boldsymbol{x}_0)\ \boldsymbol{x}) \ , \quad \forall \boldsymbol{x} \in \mathbb{R}^2 \ ,$$

where $\mathrm{D}\Phi(\boldsymbol{x}_0)$ is the jacobian matrix of $\Phi$ at point $\boldsymbol{x}_0$.

# Warped elementary field

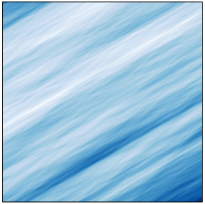
$$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$$



$X$

$$\Phi(x) \longrightarrow \mathrm{R}_{-\alpha(x)} x$$

WAFBF



$Z = X \circ \Phi$

$$\alpha(x_1, x_2) = -\frac{\pi}{4}$$

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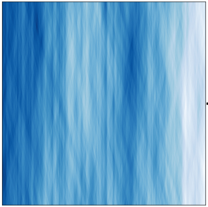
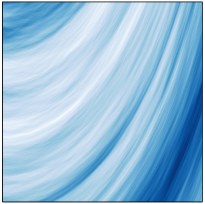
$$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$$

WAFBF



$X$

$\Phi(x)$ — $\mathrm{R}_{-\alpha(x)}x$ →



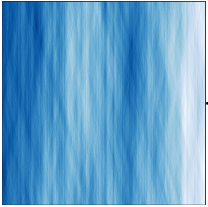
$Z = X \circ \Phi$

$$\alpha(x_1, x_2) = -\frac{\pi}{2} + x_1$$

# Warped elementary field

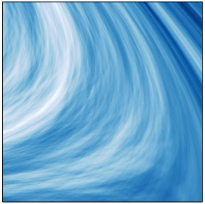
$$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$$

WAFBF



$$\Phi(x)$$

$$\mathrm{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}$$



$X$

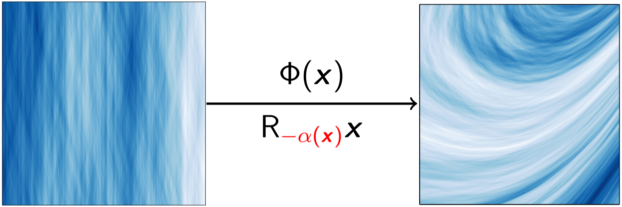
$Z = X \circ \Phi$

$$\alpha(x_1, x_2) = -\frac{\pi}{2} + x_2$$

# Warped elementary field

$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$

WAFBF



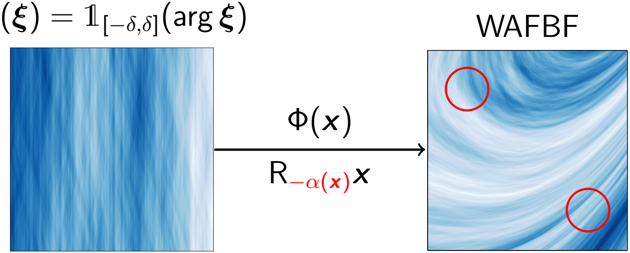

$X$

$Z = X \circ \Phi$

$$\alpha(x_1, x_2) = -\frac{\pi}{2} + x_1^2 - x_2$$

# Warped elementary field

$$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$$

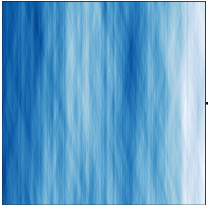



$X$ $\quad\quad$ $Z = X \circ \Phi$

1. The directionnality is not controlled

# Warped elementary field

$C(\boldsymbol{\xi}) = \mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi})$



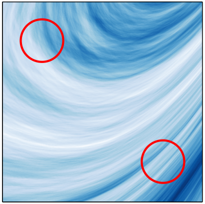
$X$

$\Phi(\boldsymbol{x})$

$\longrightarrow$

$\mathrm{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}$

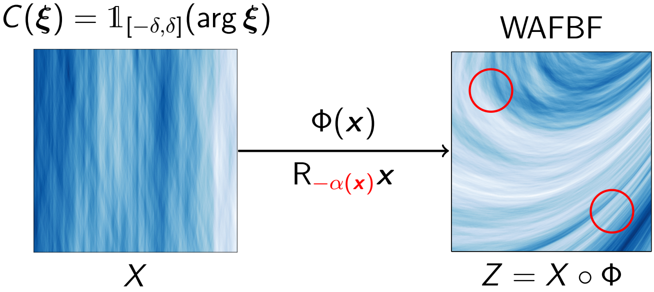
WAFBF



$Z = X \circ \Phi$

1. The directionnality is not controlled
2. What transformation $\Phi$ makes it possible to prescribe the orientation at each point $\alpha(\boldsymbol{x})$ ?

# Warped elementary field




1. The directionnality is not controlled
2. What transformation $\Phi$ makes it possible to prescribe the orientation at each point $\alpha(\mathbf{x})$ ?
3. What definition for the orientation of a random field ?

# Definition of the notion of orientation for random fields

# Local orientation of a deterministic function

## Gradient operator

The gradient operator $\nabla : f \mapsto (\partial_{x_1} f, \partial_{x_2} f)$, with the notation $\partial_{x_p} f : \boldsymbol{x} = (x_1, x_2) \mapsto \frac{\partial f}{\partial x_p}(\boldsymbol{x})$, is defined in Fourier domain by:

$$\widehat{\partial_{x_1} f}(\boldsymbol{\omega}) = -\mathrm{j}\omega_1 \widehat{f}(\boldsymbol{\omega}), \quad \widehat{\partial_{x_2} f}(\boldsymbol{\omega}) = -\mathrm{j}\omega_2 \widehat{f}(\boldsymbol{\omega})$$

$\Rightarrow$ Orientation: $\boldsymbol{n}(\boldsymbol{x}) = \dfrac{\nabla f(\boldsymbol{x})}{\|\nabla f(\boldsymbol{x})\|}$, $\theta(\boldsymbol{x}) = \arctan\left(\dfrac{\partial_{x_2} f(\boldsymbol{x})}{\partial_{x_1} f(\boldsymbol{x})}\right)$

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$\Rightarrow$ (More robust) minimize the directions against $\nabla f$:

$$\max_{\theta'} \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \left\langle \boldsymbol{n}(\theta'), \nabla f(\boldsymbol{x}') \right\rangle^2 \mathrm{d}\boldsymbol{x}' = \max_{\theta'} \boldsymbol{n}(\theta')^{\mathsf{T}} \mathrm{J}_f^W(\boldsymbol{x}) \boldsymbol{n}(\theta')$$

$$[\mathrm{J}_f^W(\boldsymbol{x})]_{pq} = \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \partial_{x_p} f(\boldsymbol{x}') \partial_{x_q} f(\boldsymbol{x}') \,\mathrm{d}\boldsymbol{x}', \quad p, q \in \{1, 2\}$$

# Local orientation of a deterministic function

## Riesz transform and monogenic signal (Felsberg, 2001)

The Riesz operator $\boldsymbol{\mathcal{R}} : f \mapsto (\mathcal{R}_1 f, \mathcal{R}_2 f)$ is defined by:

$$\widehat{\mathcal{R}_1 f}(\boldsymbol{\omega}) = -\mathrm{j}\frac{\omega_1}{\|\boldsymbol{\omega}\|}\widehat{f}(\boldsymbol{\omega}), \quad \widehat{\mathcal{R}_2 f}(\boldsymbol{\omega}) = -\mathrm{j}\frac{\omega_2}{\|\boldsymbol{\omega}\|}\widehat{f}(\boldsymbol{\omega})$$

$\Rightarrow$ Orientation: $\boldsymbol{n}(\boldsymbol{x}) = \frac{\boldsymbol{\mathcal{R}} f(\boldsymbol{x})}{\|\boldsymbol{\mathcal{R}} f(\boldsymbol{x})\|}$, $\theta(\boldsymbol{x}) = \arctan\left(\frac{\mathcal{R}_2 f(\boldsymbol{x})}{\mathcal{R}_1 f(\boldsymbol{x})}\right)$

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$f$ $\quad$ $\mathcal{R}_1 f$ $\quad$ $\mathcal{R}_2 f$ $\quad$ $\|\boldsymbol{\mathcal{R}} f\|$ $\quad$ $\theta$

Credits: R. Soulard

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$\Rightarrow$ (More robust) minimize the directions against $\boldsymbol{\mathcal{R}} f$:

$$\max_{\theta'} \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \left\langle \boldsymbol{n}(\theta'), \boldsymbol{\mathcal{R}} f(\boldsymbol{x}') \right\rangle^2 \mathrm{d}\boldsymbol{x}' = \max_{\theta'} \boldsymbol{n}(\theta')^{\mathsf{T}} \mathrm{J}_f^W(\boldsymbol{x}) \boldsymbol{n}(\theta')$$

$$[\mathrm{J}_f^W(\boldsymbol{x})]_{pq} = \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \mathcal{R}_p f(\boldsymbol{x}') \mathcal{R}_q f(\boldsymbol{x}') \, \mathrm{d}\boldsymbol{x}', \quad p, q \in \{1, 2\}$$

# Local orientation of a deterministic function

## Structure tensor

The structure tensor $J_f^w(\boldsymbol{x}) = J_f(\boldsymbol{x}) * w$ is defined from following symmetric matrix, positive definite and of rank one:

$$J_f(\boldsymbol{x}) = \boldsymbol{\mathcal{R}} f(\boldsymbol{x}) \boldsymbol{\mathcal{R}} f(\boldsymbol{x})^\mathsf{T} = \begin{pmatrix} \mathcal{R}_1 f(\boldsymbol{x})^2 & \mathcal{R}_1 f(\boldsymbol{x}) \mathcal{R}_2 f(\boldsymbol{x}) \\ \mathcal{R}_1 f(\boldsymbol{x}) \mathcal{R}_2 f(\boldsymbol{x}) & \mathcal{R}_2 f(\boldsymbol{x})^2 \end{pmatrix}$$

## Local orientation & coherency index

- The local orientation $\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{\mathcal{R}} f(\boldsymbol{x}) / \|\boldsymbol{\mathcal{R}} f(\boldsymbol{x})\|$ of $f$ at point $\boldsymbol{x}$ corresponds to the unit eigenvector associated to the largest of the eigenvalues $\lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})$ of $J_f^w(\boldsymbol{x})$
- The coherence index provides a degree of directionality:

$$\chi_f(\boldsymbol{x}) = \frac{|\lambda_2(\boldsymbol{x}) - \lambda_1(\boldsymbol{x})|}{\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})}$$

# Global definition of orientation for H-sssi fields

## Structure tensor in the self-similar stationary case

Let $X$ be a H-sssi field whose anisotropy function $C_X$ is bounded and $\psi$ a zero-mean isotropic window admitting two vanishing moments.

We define the following random structure tensor:

$$\mathrm{J}_X^{\psi} = \begin{pmatrix} |\langle X, \mathcal{R}_1\psi\rangle|^2 & \langle X, \mathcal{R}_1\psi\rangle\overline{\langle X, \mathcal{R}_2\psi\rangle} \\ \langle X, \mathcal{R}_1\psi\rangle\overline{\langle X, \mathcal{R}_2\psi\rangle} & |\langle X, \mathcal{R}_1\psi\rangle|^2 \end{pmatrix}$$

with the Gaussian variable $\langle X, \mathcal{R}_\ell\psi\rangle = \int_{\mathbb{R}^2} X(\boldsymbol{x})\mathcal{R}_\ell(\boldsymbol{x})\,\mathrm{d}x$

# Global definition of orientation for H-sssi fields

**Theorem (P. et al., 2019)**

Define $\widehat{\psi}(\boldsymbol{\xi}) = \varphi(\|\boldsymbol{\xi}\|)$. Then

$$\mathbb{E}\left[\mathrm{J}_X^{\psi}\right] = \left(\int_0^{+\infty} \frac{|\varphi(r)|^2}{r^{2H+1}}\,\mathrm{d}r\right) \mathrm{J}_X$$

where $\mathrm{J}_X$ is called the tensor structure of $X$ defined by :

$$[\mathrm{J}_X]_{\ell_1 \ell_2} = \int_{\boldsymbol{\Theta}\in\mathbb{S}^1} \Theta_{\ell_1}\Theta_{\ell_2}\; C_X(\boldsymbol{\Theta})^2\,\mathrm{d}\boldsymbol{\Theta}, \quad \ell_1, \ell_2 \in \{1, 2\}\,.$$

# Global definition of orientation for H-sssi fields

**Definition (Orientation & coherence index of a H-sssi field $X$)**

- The orientation $\vec{\boldsymbol{n}}_X$ of $X$ is given by the unit eigenvector associated to the largest of the eigenvalues $\lambda_1, \lambda_2$ of $\mathrm{J}_X$
- The coherence index of $X$ is defined by

$$\chi = \frac{|\lambda_2 - \lambda_1|}{\lambda_1 + \lambda_2}$$

# Orientations of an elementary field (EF)

## Orientation of an elementary field

$X = X_{\alpha_0,\delta}$ with $C_X(\Theta) = \mathbb{1}_{[-\delta,\delta]}(\arg\Theta - \alpha_0)$

$$\vec{\boldsymbol{n}}_X = \boldsymbol{u}(\alpha_0) = \begin{pmatrix} \cos\alpha_0 \\ \sin\alpha_0 \end{pmatrix}, \quad \chi(X) = \frac{\sin(2\delta)}{2\delta}$$

$$\mathrm{J}_X = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\cos(\alpha_0)\frac{\sin(2\delta)}{2\delta} & \frac{1}{2}\sin(\alpha_0)\frac{\sin(2\delta)}{2\delta} \\ \frac{1}{2}\sin(\alpha_0)\frac{\sin(2\delta)}{2\delta} & \frac{1}{2} - \frac{1}{2}\cos(\alpha_0)\frac{\sin(2\delta)}{2\delta} \end{pmatrix}$$

# Orientations of an elementary field (EF)

## Orientation of an elementary field

$X = X_{\alpha_0,\delta}$ with $C_X(\Theta) = \mathbb{1}_{[-\delta,\delta]}(\arg\Theta - \alpha_0)$

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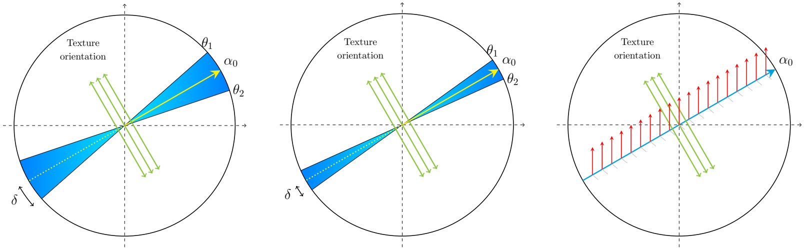


# Linear transformation of an EF and its orientation

## Sum of two independant elementary fields

Let define $X = X_{\alpha_0,\delta} + X_{\alpha_1,\delta}$ the sum of two independant EF.

$$\vec{\boldsymbol{n}}_X = \boldsymbol{u}\left(\frac{\alpha_0 + \alpha_1}{2}\right), \quad \chi(X) = \frac{\sin(2\delta)}{2\delta}\cos(\alpha_0 - \alpha_1)$$

## Deformation of an elementary field

Let $\mathsf{L}$ be an invertible $2 \times 2$ matrix and $X_{\mathsf{L}}(\boldsymbol{x}) = X_{\alpha_0,\delta}(\mathsf{L}\boldsymbol{x})$

$$\vec{\boldsymbol{n}}_{X_{\mathsf{L}}} = \frac{\mathsf{L}^{\mathsf{T}}\boldsymbol{u}(\alpha_0)}{\|\mathsf{L}^{\mathsf{T}}\boldsymbol{u}(\alpha_0)\|}$$

# Orientation of a localizable Gaussian field

## Localizable Gaussian field

A random field $X = \{X(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^2\}$ is said to be localizable, if it admits a tangent field at every point $\boldsymbol{x} \in \mathbb{R}^2$.

References : (Lévy-Véhel, 1995), (Benassi et coll., 1997), (Falconer, 2002).

## Definition (Local orientation of a localizable Gaussian field)

The local orientation $\vec{\boldsymbol{n}}_X(\boldsymbol{x}_0)$ of the localizable Gaussian field $X$ at point $\boldsymbol{x}_0$ is the orientation of its tangent field $Y_{\boldsymbol{x}_0}$ H-sssi :

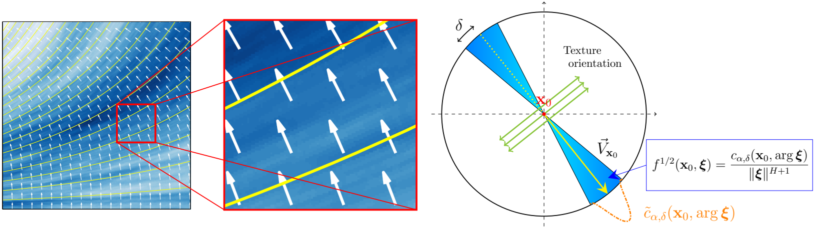
$$\vec{\boldsymbol{n}}_X(\boldsymbol{x}_0) \equiv \vec{\boldsymbol{n}}_{Y_{\boldsymbol{x}_0}}$$

# Orientation of a localizable Gaussian field

## Local orientation of the LAFBF $X$

The local orientation $\vec{n}_X(x_0)$ and the coherence index $\chi(x_0)$ of $X$ at $x_0$ are those of the elementary field $X_{\alpha(x_0),\delta(x_0)}$:

$$\vec{n}_X(x_0) \equiv \vec{n}_{X_{\alpha_0(x_0),\delta(x_0)}} = \begin{pmatrix} \cos\alpha(x_0) \\ \sin\alpha(x_0) \end{pmatrix}, \quad \chi(x_0) = \frac{\sin(2\delta(x_0))}{2\delta(x_0)}$$




(a) (b) (c)

# Orientation of a localizable Gaussian field

## Local orientation of the WAFBF where $X = X_{\alpha_0,\delta}$ is an EF

The tangent field of $Z_{\Phi,X}(\boldsymbol{x}) = X_{\alpha_0,\delta}(\Phi(\boldsymbol{x}))$ at $\boldsymbol{x}_0$ is

$$Y_{\boldsymbol{x}_0}(\boldsymbol{x}) = X_{\alpha_0,\delta}(\mathrm{D}\Phi(\boldsymbol{x}_0)\ \boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^2$$

whose orientation is $\vec{\boldsymbol{n}}_{Y_{\boldsymbol{x}_0}} = \frac{\mathrm{L}^\mathsf{T}\boldsymbol{u}(\alpha_0)}{\|\mathrm{L}^\mathsf{T}\boldsymbol{u}(\alpha_0)\|}$ with $\mathrm{L} = \mathrm{D}\Phi(\boldsymbol{x}_0)$, hence

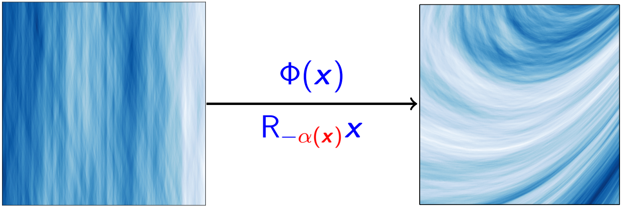
$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x}_0) \equiv \vec{\boldsymbol{n}}_{Y_{\boldsymbol{x}_0}} = \frac{\mathrm{D}\Phi(\boldsymbol{x}_0)^\mathsf{T}\boldsymbol{u}(\alpha_0)}{\|\mathrm{D}\Phi(\boldsymbol{x}_0)^\mathsf{T}\boldsymbol{u}(\alpha_0)\|}$$

# Orientation of a localizable Gaussian field

## Rotation locale du WAFBF où $X = X_{0,\delta}$

The local orientation of $Z_{\Phi_\alpha,X}(\boldsymbol{x}) = X_{0,\delta}(\Phi_\alpha(\boldsymbol{x}))$ at $\boldsymbol{x}_0$ with $\Phi_\alpha(\boldsymbol{x}) = \mathrm{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}$ is given by $\vec{\boldsymbol{n}}_Z(\boldsymbol{x}_0) = \frac{\mathrm{D}\Phi(\boldsymbol{x}_0)^\top \boldsymbol{e}_1}{\|\mathrm{D}\Phi(\boldsymbol{x}_0)^\top \boldsymbol{e}_1\|}$, that is :

$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x}_0) = \frac{\boldsymbol{u}(\alpha(\boldsymbol{x}_0)) + \langle \boldsymbol{u}(\alpha(\boldsymbol{x}_0))^\perp, \boldsymbol{x}_0\rangle \nabla\alpha(\boldsymbol{x}_0)}{\|\boldsymbol{u}(\alpha(\boldsymbol{x}_0)) + \langle \boldsymbol{u}(\alpha(\boldsymbol{x}_0))^\perp, \boldsymbol{x}_0\rangle \nabla\alpha(\boldsymbol{x}_0)\|}$$




$X_{0,\delta}$ $\qquad$ $Z_{\Phi,X} = X_{0,\delta} \circ \Phi$

# Prescribed orientations for the WAFBF

## Proposition (Orientation controled by harmonic functions)

Let $Z_{\Phi_\alpha, X}(\boldsymbol{x})$ be the field $X = X_{0,\delta}$ of orientation $\boldsymbol{e}_1 = (1,0)^{\mathsf{T}}$ warped by a conform transformation $\Phi_\alpha$ defined by:

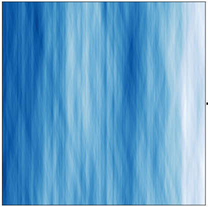
1. $\alpha : \mathbb{R}^2 \to \mathbb{R}$ a harmonic function
2. $\lambda$ its conjugate harmonic function such as $\Psi_\alpha = \begin{pmatrix} \lambda \\ -\alpha \end{pmatrix}$ is holomorphic
3. $\Phi_\alpha$ a complex primitive of $\exp(\Psi_\alpha)$

The local orientation (up to $\delta^2$) of $Z_{\Phi_\alpha, X}$ at $\boldsymbol{x}_0$ is

$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x}_0) = \begin{pmatrix} \cos\alpha(\boldsymbol{x}_0) \\ \sin\alpha(\boldsymbol{x}_0) \end{pmatrix} = \boldsymbol{u}(\alpha(\boldsymbol{x}_0))$$

# WAFBF with prescribed local orientations

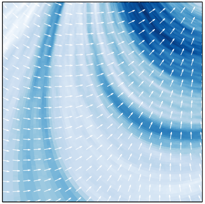
$C(\boldsymbol{\xi}) = \mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi})$



$X_{0,\delta}$

$\xrightarrow[\alpha(x_1, x_2) = ax_1 + bx_2 + c]{\Phi_\alpha}$

WAFBF $(a, b) = (2, -1)$



$Z = X_{0,\delta} \circ \Phi_\alpha$

$$\Phi_\alpha(x_1, x_2) = \frac{\exp(ax_2 - bx_1)}{a^2 + b^2} \begin{pmatrix} a \sin(ax_1 + bx_2 + c) - b \cos(ax_1 + bx_2 + c) \\ a \cos(ax_1 + bx_2 + c) + b \sin(ax_1 + bx_2 + c) \end{pmatrix}$$

$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x}) = \frac{\mathrm{D}\Phi(\boldsymbol{x})^{\mathsf{T}}(1,0)}{\|\mathrm{D}\Phi(\boldsymbol{x})^{\mathsf{T}}(1,0)\|} = (\cos \alpha(\boldsymbol{x}), \sin \alpha(\boldsymbol{x}))$$

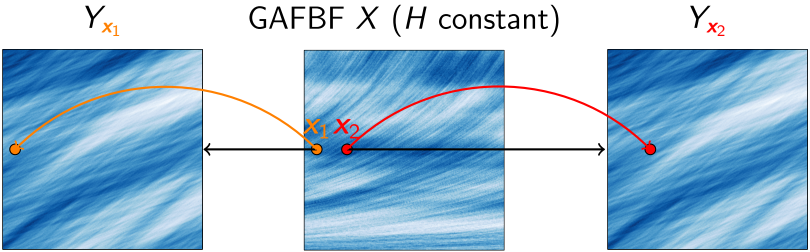
# Conclusion

- **Conclusion**
  - Two models of anisotropic Gaussian fields enabling to control the local orientation at every point
  - Definition of a global orientation for the self-similar case
  - Turn the global definition to a local one by considering localizable fields behaving locally as self-similar ones
  - Show the consistency of our approach on our two models
- **Perspectives**
  - Definition of the Riesz transform for random fields
  - Estimation of the roughness and anisotropy by wavelets
  - Test hypothesis for the directionality of a texture

# Synthesis of GAFBF inspired from (Wood, 1994)

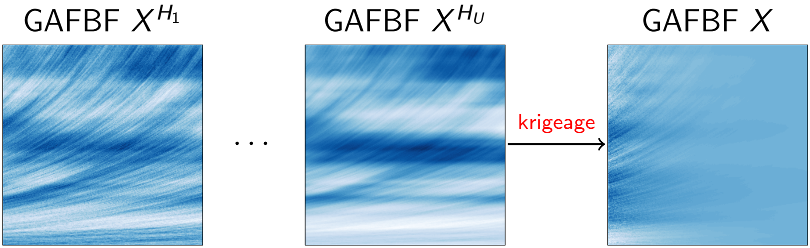
1. Simulate $U$ GAFBF $X^{H_u}$ with constant regularities $(H_u)_{1 \leq u \leq U}$ :

$$X^{H_u}(\boldsymbol{x}_0) \leftarrow Y_{\boldsymbol{x}_0}(\boldsymbol{x} = \boldsymbol{x}_0) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C_{\boldsymbol{x}_0}(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{H_u+1}} \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi})$$



# Synthesis of GAFBF inspired from (Wood, 1994)

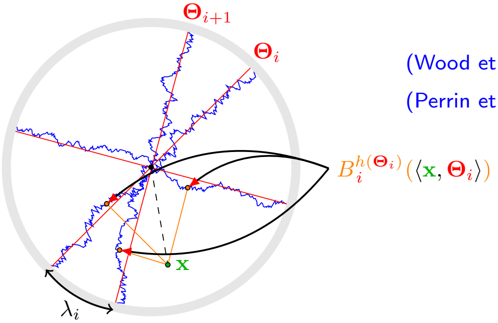
2. Simulate the GAFBF with variable regularity by krigeage : Spatial interpolation of the $(X^{H_u})$ from the covariance



# Synthesis of H-sssi fields by turning bands

$$Y_{\boldsymbol{x}_0}^{[n]}(\boldsymbol{x}) = \sum_{i=1}^{n} \omega_i(\boldsymbol{x}_0) B_i^H(\langle \boldsymbol{x}, \boldsymbol{\Theta}_i \rangle) \ ,$$

$\Rightarrow$ Simulate $n$ FBM $B_i^H$ of complexity $O(\ell \log \ell)$



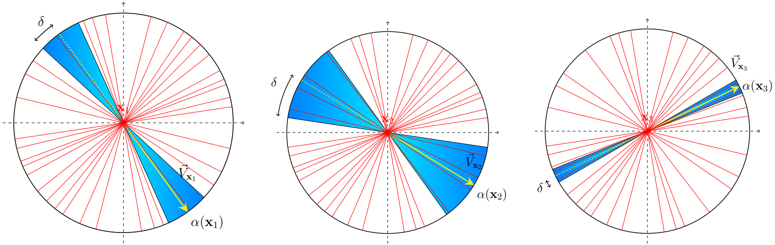

(Wood et coll., 1994)

(Perrin et coll., 2002)

# Simulation of the LAFBF with H constant

$$B^H_{\alpha,\delta}(\boldsymbol{x}_0) \leftarrow Y^{[n]}_{\boldsymbol{x}_0}(\boldsymbol{x} = \boldsymbol{x}_0) = \sum_{i=1}^{n} \omega_i(\boldsymbol{x}_0) B^H_i(\langle \boldsymbol{x}_0, \boldsymbol{\Theta}_i \rangle) \ ,$$

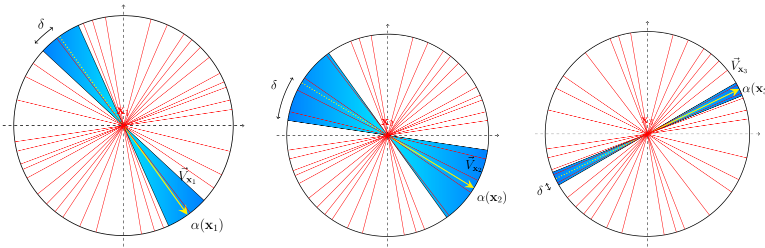
$$\omega_i(\boldsymbol{x}_0)^2 \propto C_{\boldsymbol{x}_0}(\boldsymbol{\Theta}_i) = \mathbb{1}_{[-\delta(\boldsymbol{x}_0),\delta(\boldsymbol{x}_0)]}(\arg \boldsymbol{\Theta}_i - \alpha(\boldsymbol{x}_0))$$



# Simulation of the LAFBF with H constant

$$B^H_{\alpha,\delta}(\boldsymbol{x}_0) \leftarrow Y^{[n]}_{\boldsymbol{x}_0}(\boldsymbol{x} = \boldsymbol{x}_0) = \sum_{i=1}^{n} \omega_i(\boldsymbol{x}_0) B^H_i(\langle \boldsymbol{x}_0, \boldsymbol{\Theta}_i \rangle) \ ,$$

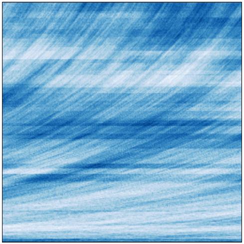
- Pre-computing of the $n$ $B^H_i$ (complexity $O(\ell \log \ell)$)
- The rest of the algorithm is of complexity $O(\log n \# \text{pixels})$



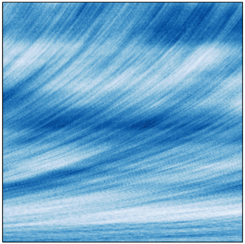
# Simulation of the LAFBF with H constant

$$B^H_{\alpha,\delta} \leftarrow Y^{[n]}_{\mathbf{x}_0}(\mathbf{x} = \mathbf{x}_0) = \sum_{i=1}^{n} \omega_i(\mathbf{x}_0) B^H_i \left(\langle \mathbf{x}_0,\, \boldsymbol{\Theta}_i \rangle\right) \;,$$

$$\omega_i(\mathbf{x}_0)^2 \propto C_{\mathbf{x}_0}(\boldsymbol{\Theta}_i) = \mathbb{1}_{[-\delta(\mathbf{x}_0),\delta(\mathbf{x}_0)]}(\arg \boldsymbol{\Theta}_i - \alpha(\mathbf{x}_0))$$



$C_{\mathbf{x}_0}(\boldsymbol{\Theta}_i)$

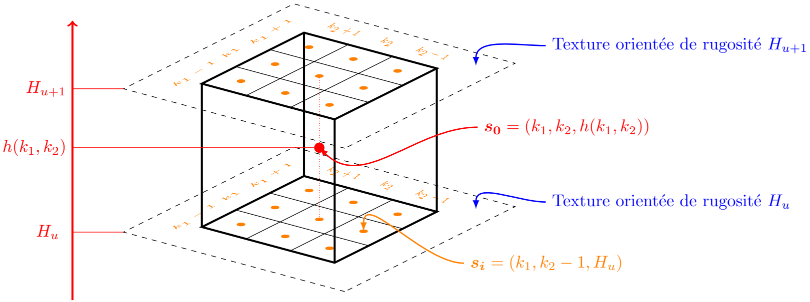


$\widetilde{C}_{\mathbf{x}_0}(\boldsymbol{\Theta}_i)$ régularisée

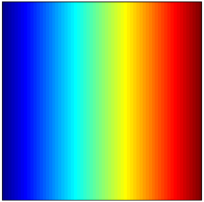
# Simulation of the LAFBF with *h* variable (krigeage)

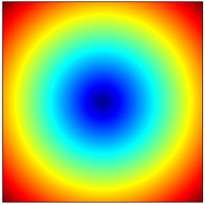
$$\widehat{Z}(\boldsymbol{s}_0) = \sum_{i \in \mathcal{V}(\boldsymbol{s}_0)} \lambda_i Z(\boldsymbol{s}_i) = \boldsymbol{\lambda}^\mathsf{T} \boldsymbol{Z} \quad (\text{BLUE})$$

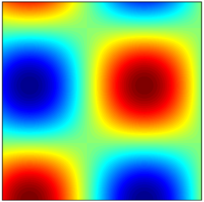
$$Z = B^h_{\alpha,\delta},\ (B^{H_u}_{\alpha,\delta})_{1 \le u \le U} \to Z(\boldsymbol{s}_i),\ \Sigma_{ij} = \mathrm{Cov}(Z(\boldsymbol{s}_i), Z(\boldsymbol{s}_j)) \to \boldsymbol{\lambda}$$

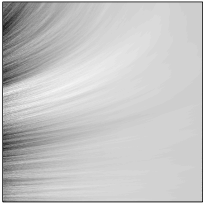


# Simulation of the LAFBF


h linear


h radial


h sinusoidal



# Local orientation of a deterministic function

## Monogenic wavelet coefficients (Unser, Olhede, 2009)

Let $\psi_{i,\mathbf{k}}(\mathbf{x}) = 2^i \psi(2^i \mathbf{x} - \mathbf{k})$ be a wavelet frame constructed from a real isotropic wavelet $\widehat{\psi}(\boldsymbol{\xi}) = \varphi(\|\boldsymbol{\xi}\|)$. We consider the wavelet coefficients of $\boldsymbol{\mathcal{R}} f$ in the frame $\{\psi_{i,\mathbf{k}}\}$:

$$c_{i,\mathbf{k}}^{(\mathcal{R})}(f) = \begin{pmatrix} c_{i,\mathbf{k}}^{(1)}(f) \\ c_{i,\mathbf{k}}^{(2)}(f) \end{pmatrix} = \begin{pmatrix} \langle \mathcal{R}_1 f, \psi_{i,\mathbf{k}} \rangle \\ \langle \mathcal{R}_2 f, \psi_{i,\mathbf{k}} \rangle \end{pmatrix} = \begin{pmatrix} \langle f, \mathcal{R}_1 \psi_{i,\mathbf{k}} \rangle \\ \langle f, \mathcal{R}_2 \psi_{i,\mathbf{k}} \rangle \end{pmatrix}$$

Tensor structure of the wavelet coefficients:

$$\mathrm{J}_{f,i}^{W}[\mathbf{k}] = c_{i,\mathbf{k}}^{(\mathcal{R})}(f) c_{i,\mathbf{k}}^{(\mathcal{R})}(f)^* = \begin{pmatrix} |c_{i,\mathbf{k}}^{(1)}(f)|^2 & c_{i,\mathbf{k}}^{(1)}(f) \cdot \overline{c_{i,\mathbf{k}}^{(2)}(f)} \\ \overline{c_{i,\mathbf{k}}^{(1)}(f)} \cdot c_{i,\mathbf{k}}^{(2)}(f) & |c_{i,\mathbf{k}}^{(1)}(f)|^2 \end{pmatrix}$$

# Orientation of a H-sssi field

## Monogenic wavelet coefficients of a H-sssi field $X$

$$c_{i,\boldsymbol{k}}^{(\ell)}(X) = \langle X, \mathcal{R}_\ell \psi_{i,\boldsymbol{k}} \rangle = \int_{\mathbb{R}^2} \widehat{\mathcal{R}_\ell \psi_{i,\boldsymbol{k}}}(\boldsymbol{\xi}) C_X(\boldsymbol{\xi}) \|\boldsymbol{\xi}\|^{-H-1} \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi})$$

## Theorem (P. et al., 2017)

Let us define $c_{i,\boldsymbol{k}}^{(\mathcal{R})}(X) = (c_{i,\boldsymbol{k}}^{(1)}(X), c_{i,\boldsymbol{k}}^{(2)}(X))^\mathsf{T}$, then:

$$\mathbb{E}[c_{i,\boldsymbol{k}}^{(\mathcal{R})}(X) c_{i,\boldsymbol{k}}^{(\mathcal{R})}(X)^*] \propto 2^{-2i(H+1)} \mathrm{J}_X \ ,$$

where $\mathrm{J}_X$ is called the tensor structure of $X$ defined by :

$$[\mathrm{J}_X]_{\ell_1 \ell_2} = \int_{\boldsymbol{\Theta} \in \mathbb{S}^1} \Theta_{\ell_1} \Theta_{\ell_2} \ C(\boldsymbol{\Theta})^2 \, \mathrm{d}\boldsymbol{\Theta}, \quad \ell_1, \ell_2 \in \{1, 2\} \ .$$