Anisotropic textures and lines within images
Analysis, synthesis and super-resolution

18 octobre 2018

Kévin Polisano

Presentation

Postdoc context

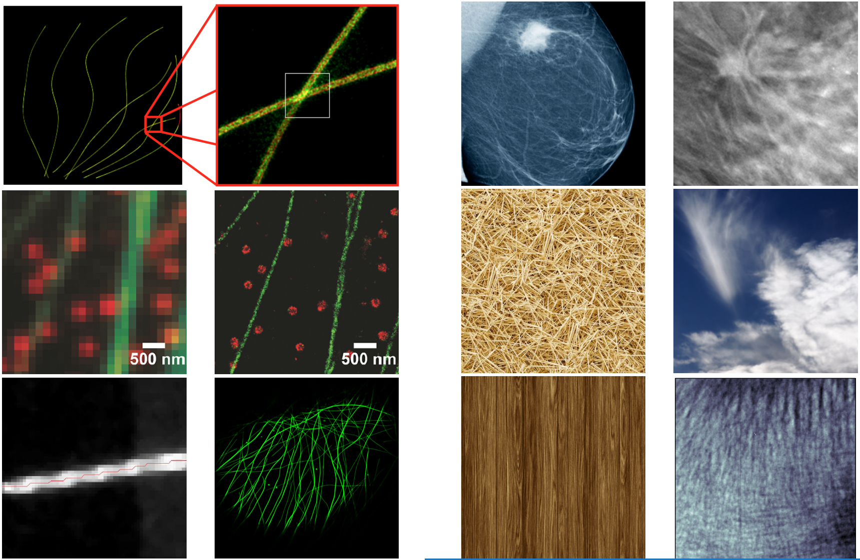
- **L**aboratoire d'**I**nformatique de **G**renoble (LIG)
- AMA group (d**A**ta analysis, **M**odeling, m**A**chine learning)
- Collaborators: Eric Gaussier (LIG, AMA)
  Adeline Leclerc-Samson (LJK, SVH)
  Jean-Marc Francony (LSS, Régulations)
- Title : "Multi-level modeling of information diffusion and opinion dynamics"
- Project of the Data Institute

Presentation

PhD context

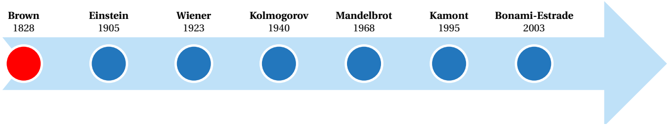
- **L**aboratoire **J**ean **K**untzmann (LJK)
- CVGI group (**C**alcul des **V**ariations, **G**éométrie, **I**mage)
- Supervisors : Valérie Perrier (Director)
  Marianne Clausel (Co-supervisor)
  Laurent Condat (Co-supervisor)
- Title : "Modeling anisotropic texture by the monogenic wavelet transform and super-resolution of 2-D lines", *supported on 2017-12-12*.
- ATER at Université Grenoble Alpes (2017–2018)

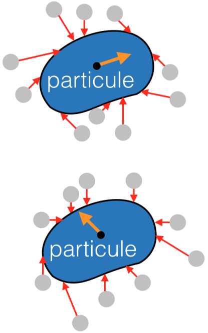
Motivations



Outline

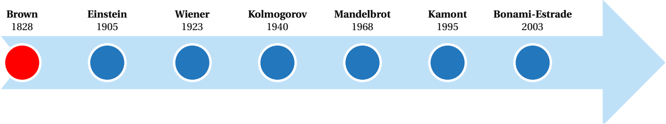
From Brownian to random anisotropic fields



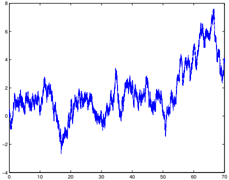


- independants displacements
- Gaussian distribution
- irregular trajectories

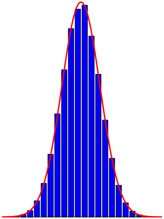
From Brownian to random anisotropic fields



independants displacements

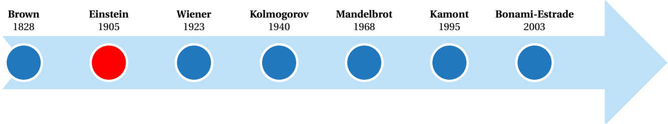


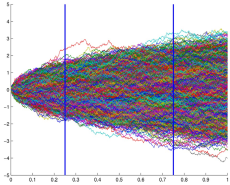
irregular trajectories

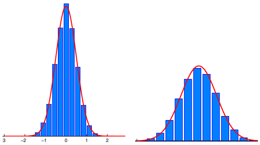


Gaussian distribution

From Brownian to random anisotropic fields

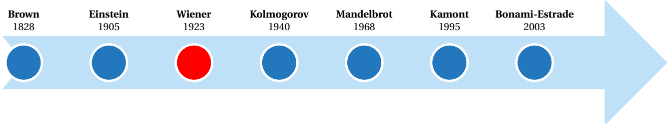






$$\overline{(\Delta x)^2} \propto t$$

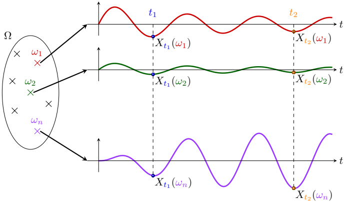
From Brownian to random anisotropic fields



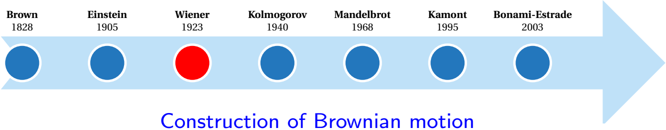
Brownian motion

- $(B_t)_t$ has independants increments, $B_0 = 0$ a.s.
- $B_{t_i} - B_{t_j} \sim \mathcal{N}(0, t_i - t_j)$
- $(B_t)_t$ has continuous sample paths a.s.

$$\begin{array}{cccc} X : & T \times \Omega & \longrightarrow & E \\ & (t,\omega) & \longmapsto & X(t,\omega) \end{array}$$



From Brownian to random anisotropic fields



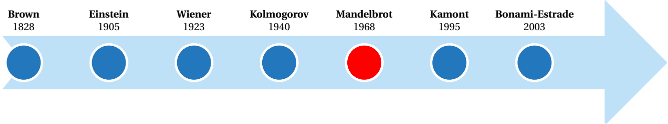
Construction of Brownian motion

Isometry $\mathbf{W} : (L^2, \langle f, g \rangle_{L^2}) \to (\mathcal{G}, \mathbb{E}[XY])$

- $\mathbb{E}[\mathbf{W}(f)\mathbf{W}(g)] = \langle f, g \rangle_{L^2}, \quad \mathbf{W}(f) \sim \mathcal{N}(0, \|f\|_{L^2}^2)$
- $\forall t \in [0,1], \quad B_t \stackrel{\text{def}}{=} \mathbf{W}(\mathbb{1}_{[0,t]})$
  - $\mathbb{E}\left[(B_t - B_s)^2\right] = \|\mathbb{1}_{[0,t]} - \mathbb{1}_{[0,s]}\|_{L^2}^2 = \int \mathbb{1}_{[s,t]} = t - s$
  - $\mathbb{E}\left[(B_{t_i} - B_{t_i-1})(B_{t_j} - B_{t_j-1})\right] = \langle \mathbb{1}_{[t_{i-1},t_i]}, \mathbb{1}_{[t_{j-1},t_j]} \rangle_{L^2} = 0$

$$\text{Wiener stochastic integral} = \int f(x)\mathbf{W}(\mathrm{d}x)$$

From Brownian to random anisotropic fields

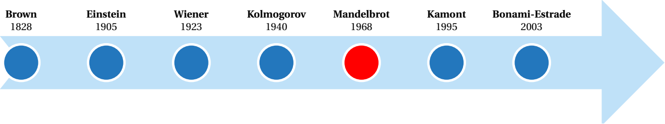


Self-similarity

$\{X(t)\}_{t\in T}$ is self-similar of order H if $\forall\lambda \in \mathbb{R}$

$$\{X(\lambda t)\}_{t\in T} \overset{(fdd)}{=} \lambda^H \{X(t)\}_{t\in T}$$

From Brownian to random anisotropic fields

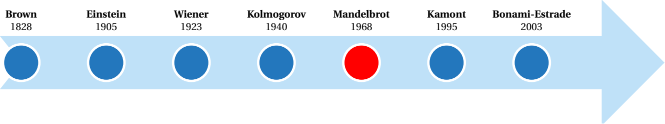


Self-similarity

$\{X(t)\}_{t\in T}$ is self-similar of order H if $\forall\lambda \in \mathbb{R}$

$$\{X(\lambda t)\}_{t\in T} \overset{(fdd)}{=} \lambda^H \{X(t)\}_{t\in T}$$

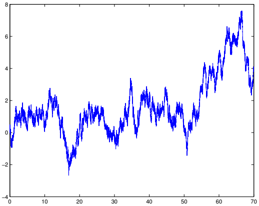
From Brownian to random anisotropic fields



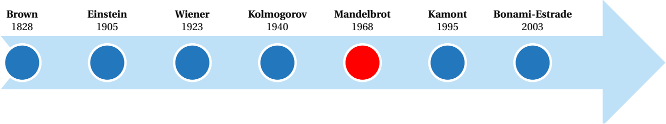
Self-similarity

$\{X(t)\}_{t\in T}$ is self-similar of order H if $\forall\lambda \in \mathbb{R}$

$$\{X(\lambda t)\}_{t\in T} \stackrel{(fdd)}{=} \lambda^H \{X(t)\}_{t\in T}$$

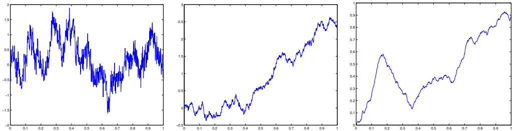


From Brownian to random anisotropic fields



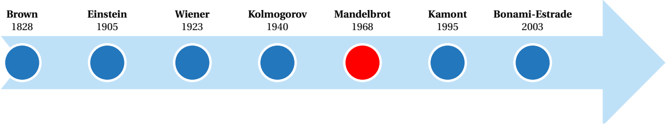
- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ ~~indpt. increments~~

fractional Brownian motion B^H



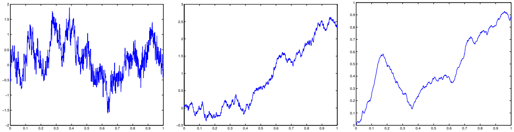
$H = 0.2$ $H = 0.5$ $H = 0.8$

From Brownian to random anisotropic fields



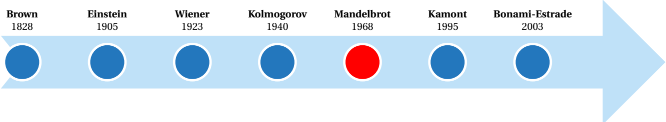
- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ stat. increments

fractional Brownian motion B^H (FBM)



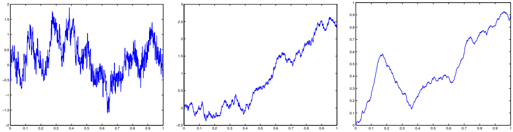
$H = 0.2$ $H = 0.5$ $H = 0.8$

From Brownian to random anisotropic fields



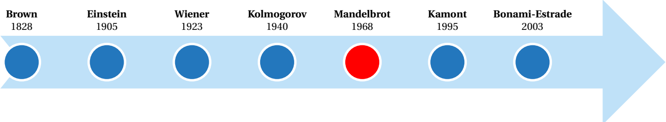
- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ stat. increments
- $\mathbf{R}(t,s) = \mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$

fractional Brownian motion B^H (FBM)



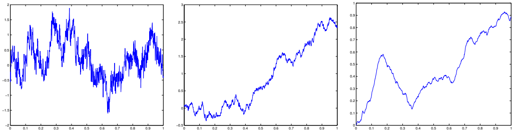
$H = 0.2$ $H = 0.5$ $H = 0.8$

From Brownian to random anisotropic fields



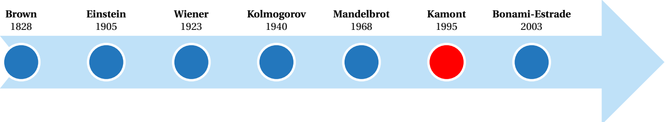
- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ stat. increments
- $\mathbf{R}(t,s) = \mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$
- $B^H(t) = \frac{1}{c_H} \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{j}t\xi} - 1}{|\xi|^{H+1/2}} \widehat{\mathbf{W}}(\xi) \Rightarrow$ harmonizable formula

fractional Brownian motion B^H (FBM)



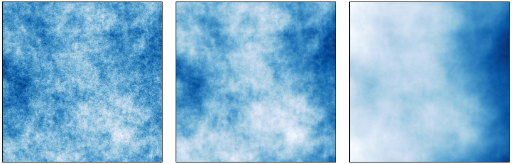
$H = 0.2$ $\quad$ $H = 0.5$ $\quad$ $H = 0.8$

From Brownian to random anisotropic fields



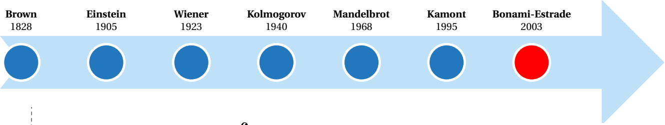
- $\mathbb{E}\left[(B^H(\boldsymbol{x}) - B^H(\boldsymbol{y}))^2\right] = \|\boldsymbol{x} - \boldsymbol{y}\|^{2H}$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$
- $\mathbf{R}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\left(\|\boldsymbol{x}\|^{2H} + \|\boldsymbol{y}\|^{2H} - \|\boldsymbol{x} - \boldsymbol{y}\|^{2H}\right)$
- $B^H(\boldsymbol{x}) = \frac{1}{C_H} \int_{\mathbb{R}^2} \frac{\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1}{\|\boldsymbol{\xi}\|^{H+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$

fractional Brownian field B^H (FBF)

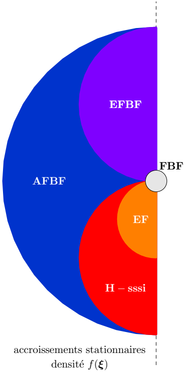


$H = 0.2$ $H = 0.5$ $H = 0.8$

Modèle de Bonami-Estrade

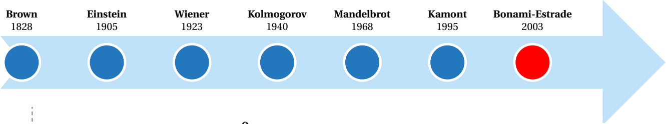


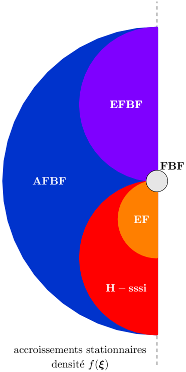
$$X(x) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{\xi}) \, \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$



- $f^{1/2}(\boldsymbol{\xi}) = \dfrac{C}{\|\boldsymbol{\xi}\|^{H+1}}$ (FBF)
- $f^{1/2}(\boldsymbol{\xi}) = \dfrac{C}{\|\boldsymbol{\xi}\|^{H+1}}$ (FBF)
- $f^{1/2}(\boldsymbol{\xi}) = \dfrac{C}{\|\boldsymbol{\xi}\|^{H+1}}$ (FBF)

Model of Bonami-Estrade

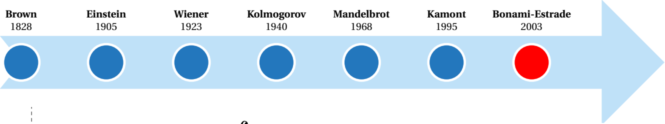


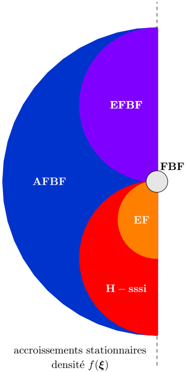


$$X(x) = \int_{\mathbb{R}^2} (e^{j\langle x, \xi\rangle} - 1) f^{1/2}(\xi)\, \widehat{W}(d\xi)$$

- $f^{1/2}(\xi) = \dfrac{C}{\|\xi\|^{h(\xi)+1}}$ (EFBF)
- $f^{1/2}(\xi) = \dfrac{C}{\|\xi\|^{H+1}}$ (FBF)
- $f^{1/2}(\xi) = \dfrac{C}{\|\xi\|^{H+1}}$ (FBF)

Model of Bonami-Estrade

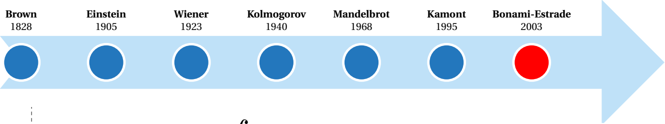


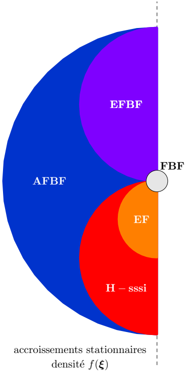


$$X(x) = \int_{\mathbb{R}^2} (e^{j\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1) f^{1/2}(\boldsymbol{\xi}) \widehat{\mathbf{W}}(d\boldsymbol{\xi})$$

- $f^{1/2}(\boldsymbol{\xi}) = \dfrac{C}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{\xi})+1}}$ (EFBF)
- $f^{1/2}(\boldsymbol{\xi}) = \dfrac{C(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{H+1}}$ (H-sssi) (Benassi et coll., 1997)
- $f^{1/2}(\boldsymbol{\xi}) = \dfrac{C}{\|\boldsymbol{\xi}\|^{H+1}}$ (FBF)

Model of Bonami-Estrade

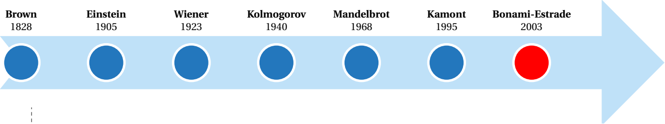


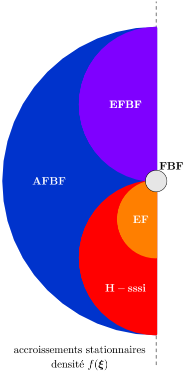


$$X(x) = \int_{\mathbb{R}^2} (e^{j\langle x, \xi \rangle} - 1) f^{1/2}(\xi) \, \widehat{W}(d\xi)$$

- $f^{1/2}(\xi) = \frac{C}{\|\xi\|^{h(\xi)+1}}$ (EFBF)
- $f^{1/2}(\xi) = \frac{C(\xi)}{\|\xi\|^{H+1}}$ (H-sssi) (Benassi et coll., 1997)
- $f^{1/2}(\xi) = \frac{C(\xi)}{\|\xi\|^{h(\xi)+1}}$ (AFBF)

Model of Bonami-Estrade



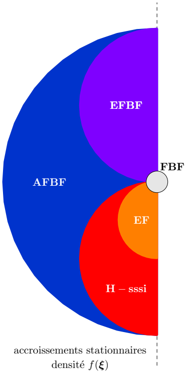


$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{\xi}) \, \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

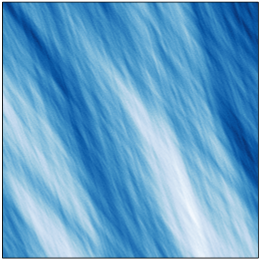
- $f^{1/2}(\boldsymbol{\xi}) = \dfrac{C}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{\xi})+1}}$ (EFBF)
- $f^{1/2}(\boldsymbol{\xi}) = \dfrac{C(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{H+1}}$ (H-sssi) (Benassi et coll., 1997)
  - $C(\boldsymbol{\xi}) = \mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha_0)$ (EF)

Model of Bonami-Estrade

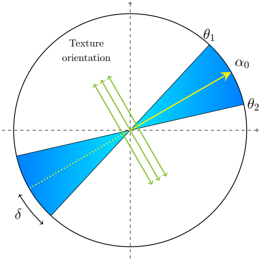
$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1\right) \frac{\mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha_0)}{\|\boldsymbol{\xi}\|^{H+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$



Elementary field (EF) $[H = 0.5,\ \alpha_0 = \pi/6]$



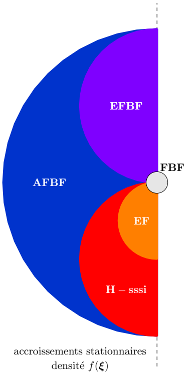
$\delta = 3.10^{-1}$



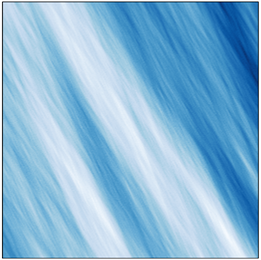
$\delta = 3.10^{-1}$

Model of Bonami-Estrade

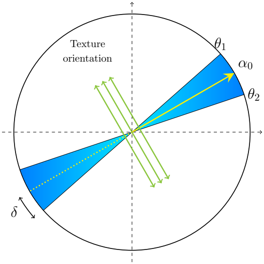
$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(e^{j\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1\right) \frac{\mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha_0)}{\|\boldsymbol{\xi}\|^{H+1}} \widehat{\mathbf{W}}(d\boldsymbol{\xi})$$



Elementary field (EF) $[H = 0.5,\ \alpha_0 = \pi/6]$



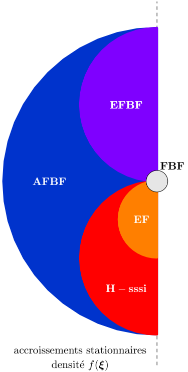
$\delta = 3.10^{-1}$



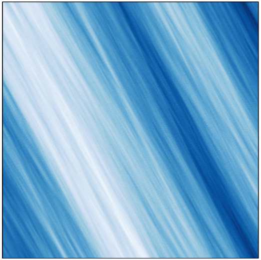
$\delta = 3.10^{-1}$

Model of Bonami-Estrade

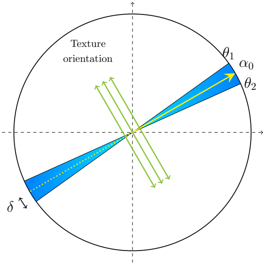
$$X(x) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle x, \xi \rangle} - 1\right) \frac{\mathbb{1}_{[-\delta,\delta]}(\arg \xi - \alpha_0)}{\|\xi\|^{H+1}} \widehat{\mathbf{W}}(\mathrm{d}\xi)$$



Elementary field (EF) $[H = 0.5,\ \alpha_0 = \pi/6]$



$\delta = 3.10^{-1}$



$\delta = 3.10^{-1}$

State of the art : anisotropic Gaussian fields

- Fractional Brownian sheet (FBS) (Kamont, 1995), (Léger and Pontier, 1999), (Ayache et al., 2002)
- H-sssi fields (Benassi et coll., 1997)
- Model of Bonami and Estrade (Bonami and Estrade, 2003)
- Operator scaling Gaussian random fields (OSGRF) (Schertzer and Lovejoy, 1985), (Biermé et. al, 2007)
- Model of Xue, Xiao, Li (Xue and Xiao, 2011), (Li and Xiao, 2011)
- ...

State of the art : anisotropic Gaussian fields

- Fractional Brownian sheet (FBS) (Kamont, 1995), (Léger and Pontier, 1999), (Ayache et al., 2002)
- H-sssi fields (Benassi et coll., 1997)
- Model of Bonami and Estrade (Bonami and Estrade, 2003)
- Operator scaling Gaussian random fields (OSGRF) (Schertzer and Lovejoy, 1985), (Biermé et. al, 2007)
- Model of Xue, Xiao, Li (Xue and Xiao, 2011), (Li and Xiao, 2011)
- ...

$\Rightarrow$ no class of fields with controlled local anisotropy

State of the art : anisotropic Gaussian fields

- Fractional Brownian sheet (FBS) (Kamont, 1995), (Léger and Pontier, 1999), (Ayache et al., 2002)
- H-sssi fields (Benassi et coll., 1997)
- Model of Bonami and Estrade (Bonami and Estrade, 2003)
- Operator scaling Gaussian random fields (OSGRF) (Schertzer and Lovejoy, 1985), (Biermé et. al, 2007)
- Model of Xue, Xiao, Li (Xue and Xiao, 2011), (Li and Xiao, 2011)
- ...

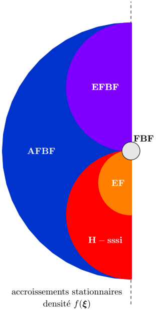
$\Rightarrow$ no class of fields with controlled local anisotropy

$\Rightarrow$ contribution : two new classes of this type the (GAFBF) and the (WAFBF)

From H-sssi fields to GAFBF

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{\xi}) \, \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

If X is H–self-similar, that is $X(\lambda \boldsymbol{x}) = \lambda^H X(\boldsymbol{x})$, one has:



H-sssi

$$f^{1/2}(\boldsymbol{\xi}) = \frac{C(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{H+1}}$$

EF

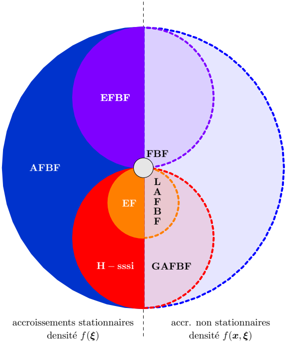
$$C(\boldsymbol{\xi}) = \mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha_0)$$

with homogeneous anisotropic function $\boldsymbol{\xi} \mapsto C(\boldsymbol{\xi})$

Model with prescribed orientations and regularities

New model: a localized and multifractional version of H-sssi fields

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1\right) f^{1/2}(\boldsymbol{x}, \boldsymbol{\xi}) \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$



GAFBF

$$f^{1/2}(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{C(\boldsymbol{x}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x})+1}}$$

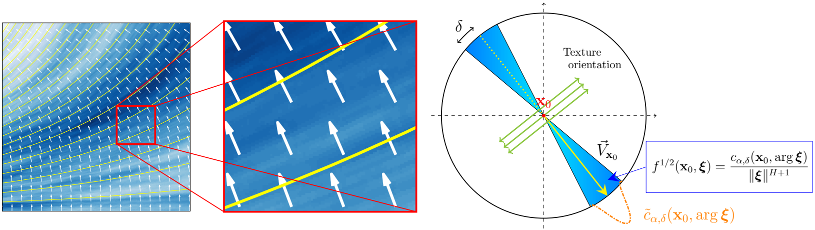
LAFBF

$$C(\boldsymbol{x}, \boldsymbol{\xi}) = \mathbb{1}_{[-\delta(\boldsymbol{x}), \delta(\boldsymbol{x})]}(\arg \boldsymbol{\xi} - \alpha(\boldsymbol{x}))$$

Model with prescribed local orientation

$$B^H_{\alpha,\delta}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(e^{j\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1\right) \frac{\mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha(\boldsymbol{x}))}{\|\boldsymbol{\xi}\|^{H+1}} \widehat{\mathbf{W}}(d\boldsymbol{\xi})$$

localized elementary field (LAFBF) $[H = 0.8,\ \alpha(x_1, x_2) = -\pi/2 + x_1]$



(a) (b) (c)

The tangent field: a tool for analysis and synthesis

1. A tool for analysis (Lévy-Vehel, 1995), (Falconer, 2002) :

$$\left\{ \lim_{\rho \to 0} \frac{X(\mathbf{x_0} + \rho \boldsymbol{x}) - X(\mathbf{x_0})}{\rho^{h(\boldsymbol{x}_0)}} \right\}_{\boldsymbol{x} \in \mathbb{R}^2} \stackrel{d}{=} \{ Y_{\boldsymbol{x}_0}(\boldsymbol{x}) \}_{\boldsymbol{x} \in \mathbb{R}^2}$$

Roughly speaking $Y_{\boldsymbol{x}_0}$ is the "local form" of X at point $\boldsymbol{x}_0$.

2. A tool for synthesis (Lévy-Véhel, 1995), (Benassi, 1997) :

$$X(\boldsymbol{x}_0) \leftarrow Y_{\boldsymbol{x}_0}(\boldsymbol{x} = \boldsymbol{x}_0)$$

$\Rightarrow$ If Y is "localizable", all local anisotropy characteristics are defined and herited from its tangent field.

Assumptions of the GAFBF

Assumptions $(\mathcal{H})$

- h is β–Hölder, such that $a = \inf_{\boldsymbol{x}\in\mathbb{R}^2} h(\boldsymbol{x}) > 0$, $b = \sup_{\boldsymbol{x}\in\mathbb{R}^2} h(\boldsymbol{x})$ and $b < \beta \leq 1$.
- $(\boldsymbol{x},\boldsymbol{\xi}) \mapsto C(\boldsymbol{x},\boldsymbol{\xi})$ is bounded $C(\boldsymbol{x},\boldsymbol{\xi}) \leqslant M, \forall(\boldsymbol{x},\boldsymbol{\xi})$.
- $\boldsymbol{\xi} \mapsto C(\boldsymbol{x},\boldsymbol{\xi})$ is even $C(\boldsymbol{x},-\boldsymbol{\xi}) = C(\boldsymbol{x},\boldsymbol{\xi})$.
- $\boldsymbol{\xi} \mapsto C(\boldsymbol{x},\boldsymbol{\xi})$ homogeneous $C(\boldsymbol{x},\rho\boldsymbol{\xi}) = C(\boldsymbol{x},\boldsymbol{\xi}), \forall\rho$.
- $\boldsymbol{x} \mapsto C(\boldsymbol{x},\boldsymbol{\xi})$ is continuous and $\exists\eta,\ \beta \leq \eta \leq 1,\ \forall\boldsymbol{x}$

$$\sup_{\boldsymbol{z}\in B(\boldsymbol{0},1)} \|\boldsymbol{z}\|^{-2\eta} \int_{\mathbb{S}^1} \left[C(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{\Theta}) - C(\boldsymbol{x},\boldsymbol{\Theta})\right]^2 \mathrm{d}\boldsymbol{\Theta} \leq A_{\boldsymbol{x}} < \infty$$

Tangent field of the GAFBF

Let X be the GAFBF defined by

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1\right) \frac{C(\boldsymbol{x}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x})+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

Theorem (Polisano et coll., 2017)

If X satisfies the assumptions $(\mathcal{H})$, then X admits in every point $\boldsymbol{x_0} \in \mathbb{R}^2$ a tangent field $Y_{\boldsymbol{x_0}}$ given by:

$$\begin{aligned} Y_{\boldsymbol{x_0}}(\boldsymbol{x}) &= \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1) f^{1/2}(\boldsymbol{x_0}, \xi) \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi}) \ , \\ &= \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1) \frac{C(\boldsymbol{x_0}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x_0})+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi}) \ . \end{aligned}$$

Tangent field of the GAFBF

Let X be the GAFBF defined by

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1\right) \frac{C(\boldsymbol{x}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x})+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

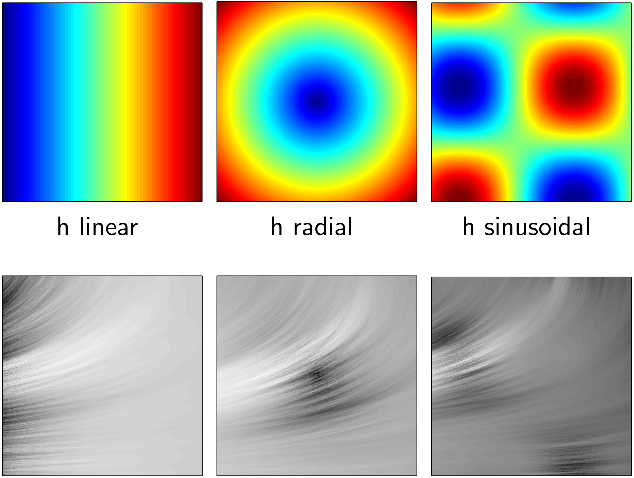
Theorem (Polisano et coll., 2017)

If X satisfies the assumptions $(\mathcal{H})$, then X admits in every point $\boldsymbol{x_0} \in \mathbb{R}^2$ a tangent field $Y_{\boldsymbol{x_0}}$ given by:

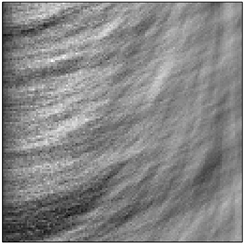
$$Y_{\boldsymbol{x_0}}(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{x_0}, \boldsymbol{\xi}) \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi}) \, ,$$

$$\text{H-sssi field} = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C_{\boldsymbol{x_0}}(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x_0})+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi}) \, .$$

Simulation of the LAFBF



Simulation of the LAFBF



- Linear variation of the orientations $\alpha(\boldsymbol{x})$ along (Ox)
- Linear variation of the directionality $\delta(\boldsymbol{x})$ along (Ox)
- Linear variation of the regularity $h(\boldsymbol{x})$ along (Ox)

Local orientation of a deterministic function

Gradient operator

The gradient operator $\nabla : f \mapsto (\partial_{x_1} f, \partial_{x_2} f)$, with the notation $\partial_{x_1} f : \boldsymbol{x} = (x_1, x_2) \mapsto \frac{\partial f}{\partial x_1}(\boldsymbol{x})$, is defined in Fourier domain by:

$$\widehat{\partial_{x_1} f}(\boldsymbol{\omega}) = -\mathrm{j}\omega_1 \widehat{f}(\boldsymbol{\omega}), \quad \widehat{\partial_{x_2} f}(\boldsymbol{\omega}) = -\mathrm{j}\omega_2 \widehat{f}(\boldsymbol{\omega})$$

$\Rightarrow$ Orientation :

$$\boldsymbol{n}(\boldsymbol{x}) = \frac{\nabla f(\boldsymbol{x})}{\|\nabla f(\boldsymbol{x})\|}$$

Local orientation of a deterministic function

Riesz transform and monogenic signal (Felsberg, 2001)

The Riesz operator $\boldsymbol{\mathcal{R}} : f \mapsto (\mathcal{R}_1 f, \mathcal{R}_2 f)$ is defined by:

$$\widehat{\mathcal{R}_1 f}(\boldsymbol{\omega}) = -\mathrm{j}\frac{\omega_1}{\|\boldsymbol{\omega}\|}\widehat{f}(\boldsymbol{\omega}), \quad \widehat{\mathcal{R}_2 f}(\boldsymbol{\omega}) = -\mathrm{j}\frac{\omega_2}{\|\boldsymbol{\omega}\|}\widehat{f}(\boldsymbol{\omega})$$

$\Rightarrow$ Orientation :

$$\boldsymbol{n}(\boldsymbol{x}) = \frac{\boldsymbol{\mathcal{R}} f(\boldsymbol{x})}{\|\boldsymbol{\mathcal{R}} f(\boldsymbol{x})\|}$$

Orientation of a H-sssi field

Monogenic wavelet coefficients of a H-sssi field X

$$c_{i,\mathbf{k}}^{(\ell)}(X) = \langle X, \mathcal{R}_\ell \psi_{i,\mathbf{k}} \rangle = \int_{\mathbb{R}^2} \widehat{\mathcal{R}_\ell \psi_{i,\mathbf{k}}}(\boldsymbol{\xi}) C(\boldsymbol{\xi}) \|\boldsymbol{\xi}\|^{-H-1} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

Theorem (Polisano et al., 2017)

Let us define $c_{i,\mathbf{k}}^{(\mathcal{R})}(X) = (c_{i,\mathbf{k}}^{(1)}(X), c_{i,\mathbf{k}}^{(2)}(X))^\mathsf{T}$, then:

$$\mathbb{E}[c_{i,\mathbf{k}}^{(\mathcal{R})}(X) c_{i,\mathbf{k}}^{(\mathcal{R})}(X)^*] \propto 2^{-2i(H+1)} \mathbf{J}_X \ ,$$

where $\mathbf{J}_X$ is called the tensor structure of X defined by :

$$[\mathbf{J}_X]_{\ell_1 \ell_2} = \int_{\boldsymbol{\Theta} \in \mathbb{S}^1} \Theta_{\ell_1} \Theta_{\ell_2} \ C(\boldsymbol{\Theta})^2 \,\mathrm{d}\boldsymbol{\Theta}, \quad \ell_1, \ell_2 \in \{1, 2\} \ .$$

Orientation of a H-sssi field

Definition (Orientation and coherence index of a H-sssi field)

- The orientation $\vec{\boldsymbol{n}}_X$ of X is given by the unit eigenvector associated to the largest of the eigenvalues λ_1, λ_2 of $\mathbf{J}_X$
- The coherence index of X is defined by

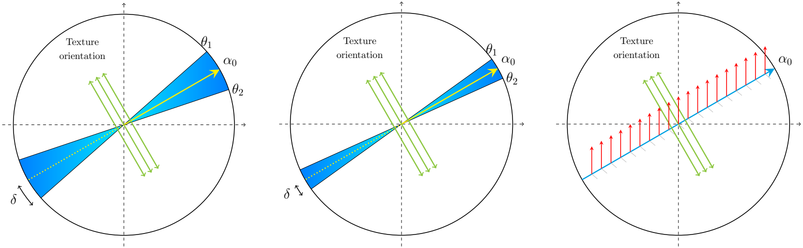
$$\chi = \frac{|\lambda_2 - \lambda_1|}{\lambda_1 + \lambda_2}$$

Orientations of an elementary field

Exemple (Orientation of an EF)

$X = X_{\alpha_0,\delta}$ with $C(\boldsymbol{\Theta}) = \mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\Theta} - \alpha_0)$

$$\vec{\boldsymbol{n}}_X = \begin{pmatrix} \cos \alpha_0 \\ \sin \alpha_0 \end{pmatrix}, \quad \chi = \frac{\sin(2\delta)}{2\delta}$$



Orientation of a localizable Gaussian field

Localizable Gaussian field

A random field $X = \{X(\boldsymbol{x}),\ \boldsymbol{x} \in \mathbb{R}^2\}$ is said to be localizable, if it admits a tangent field at every point $\boldsymbol{x} \in \mathbb{R}^2$.

References : (Lévy-Véhel, 1995), (Benassi et coll., 1997), (Falconer, 2002).

Definition (Local orientation of a localizable Gaussian field)

The local orientation $\vec{\boldsymbol{n}}_X(\boldsymbol{x}_0)$ of the localizable Gaussian field X at point $\boldsymbol{x_0}$ is the orientation of its tangent field $Y_{\boldsymbol{x_0}}$ H-sssi :

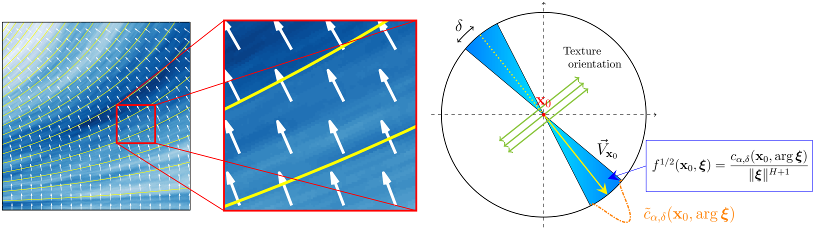
$$\vec{\boldsymbol{n}}_X(\boldsymbol{x}_0) \equiv \vec{\boldsymbol{n}}_{Y_{\boldsymbol{x}_0}}$$

Orientation of a localizable Gaussian field

Local orientation of the LAFBF X

The local orientation $\vec{n}_X(x_0)$ and the coherence index $\chi(x_0)$ of X at x_0 are those of the elementary field $X_{\alpha(x_0),\delta(x_0)}$:

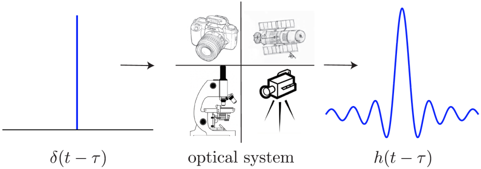
$$\vec{n}_X(x_0) \equiv \vec{n}_{X_{\alpha_0(x_0),\delta(x_0)}} = \begin{pmatrix} \cos\alpha(x_0) \\ \sin\alpha(x_0) \end{pmatrix}, \quad \chi(x_0) = \frac{\sin(2\delta(x_0))}{2\delta(x_0)}$$



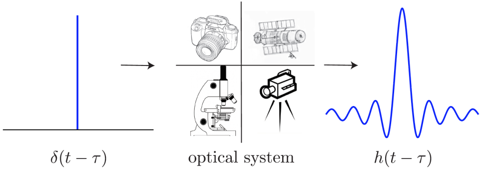
(a) (b) (c)

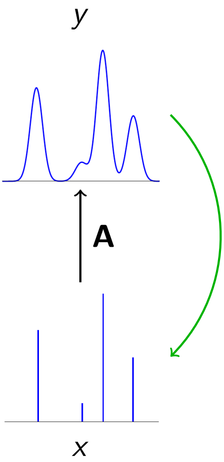
Outline

Diffraction and Rayleigh limit



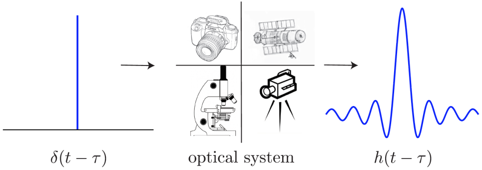
Super-resolution of 1-D impulses

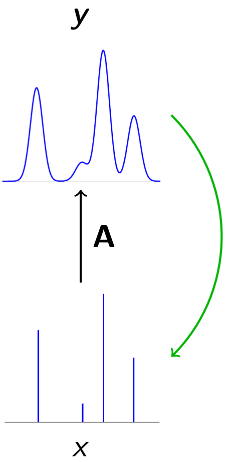




$$x = \sum_{i=1}^{K} c_i \delta_{t_i}, \quad c_i \geq 0, \quad t_i \geq 0$$

Super-resolution of 1-D impulses

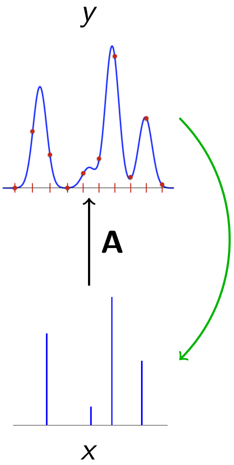




$$y(t) = \sum_{i=1}^{K} c_i h(t - t_i), \quad c_i \geq 0, \quad t_i \geq 0$$

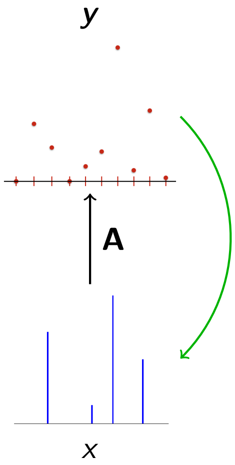
Discrete data on a grid

$$\boldsymbol{y} = y(\tau_k), \quad \tau_k = k\Delta/N$$



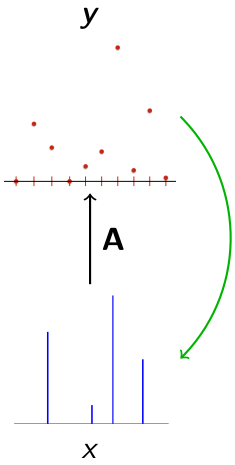
Discrete data on a grid

$$\mathbf{y} = y(\tau_k), \quad \tau_k = k\Delta/N$$



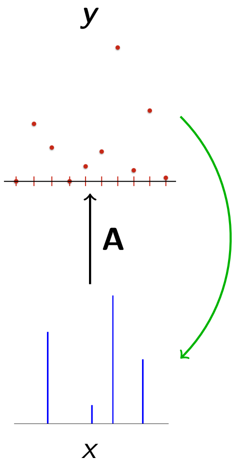
Parcimonious reconstruction on the grid

$$\min_{\boldsymbol{c}\in\mathbb{R}^K} \frac{1}{2}\left\|\boldsymbol{y}-\mathbf{A}\boldsymbol{c}\right\|_2^2+\lambda\left\|\boldsymbol{c}\right\|_0$$



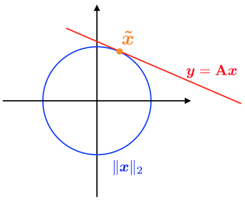
Parcimonious convex reconstruction on the grid

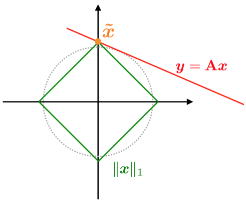
$$\min_{\boldsymbol{c}\in\mathbb{R}^K} \frac{1}{2}\|\boldsymbol{y}-\mathbf{A}\boldsymbol{c}\|_2^2+\lambda\|\boldsymbol{c}\|_1$$

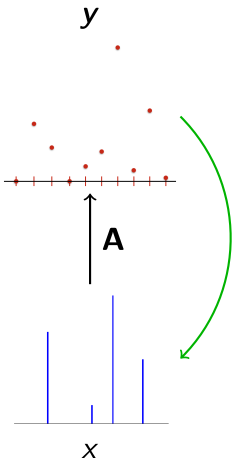


Parcimonious convex reconstruction on the grid

$$\min_{c \in \mathbb{R}^K} \frac{1}{2} \|y - \mathbf{A}c\|_2^2 + \lambda \|c\|_1$$



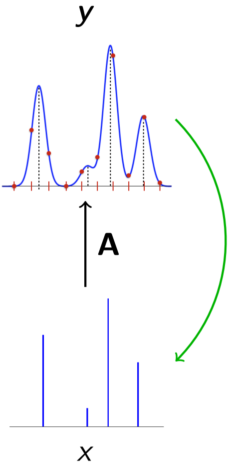




Super-resolution of 1-D impulses on a grid

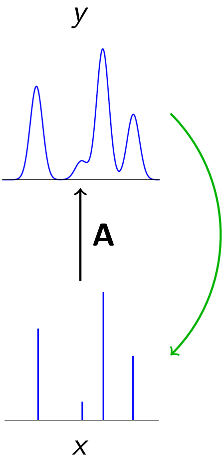
$$\min_{\boldsymbol{c}\in\mathbb{R}^K} \frac{1}{2}\|\boldsymbol{y}-\mathbf{A}\boldsymbol{c}\|_2^2+\lambda\|\boldsymbol{c}\|_1$$

$$\boldsymbol{y}=y(\tau_k), \quad \tau_k=k\Delta/N \longrightarrow \tilde{x}_k$$



Super-resolution of 1-D impulses off-the-grid

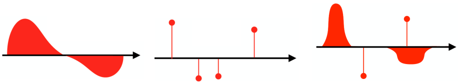
$$x = \sum_{i=1}^{K} c_i \delta_{t_i}, \quad c_i \geq 0, \quad t_i \geq 0$$



Minimization (convex regularization)

$$\arg\min_{\mu} \frac{1}{2} \|y - \mathbf{A}\mu\|^2 + \lambda \|\mu\|_{\mathrm{TV}}$$

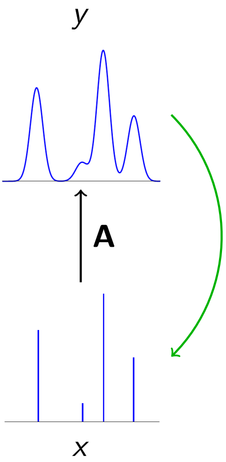
Reference : (Candès, Fernandez-Granda, 2012)



$\|\mu\|_{\mathrm{TV}} = \int |f|$ $\|x\|_{\mathrm{TV}} = \|\mathbf{c}\|_1$

Super-resolution of 1-D impulses off-the-grid

$$(\mathcal{F}x)(\omega) = \sum_{i=1}^{K} c_i \mathrm{e}^{j2\pi f_i \omega}, \quad c_i \geq 0, \quad t_i \geq 0$$



Minimization (convex regularization)

$$\arg\min_x \frac{1}{2} \|y - \mathbf{A}x\|^2 + \lambda \|x\|_{\mathrm{TV}}$$

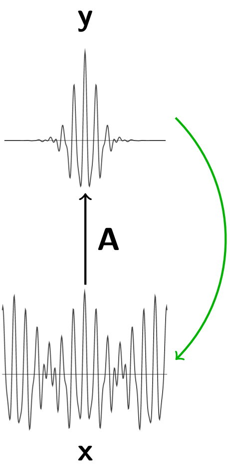
Reference : (Tang, Bhaskar, Recht et al., 2013)

Super-resolution of 1-D impulses off-the-grid

$$\mathbf{x} = \sum_{i=1}^{K} c_i \mathbf{a}(f_i), \quad c_i \geq 0, \quad \mathbf{a}(f_i) \in \mathcal{A}$$

$$\mathcal{A} = \left\{ \boldsymbol{a}(f) \in \mathbb{C}^N \right\}, \quad [\boldsymbol{a}(f)]_n = \mathrm{e}^{\mathrm{j}2\pi f n}$$

$$\|\mathbf{x}\|_{\mathcal{A}} = \inf \left\{ \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} : \mathbf{x} = \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} \mathbf{a} \right\}$$



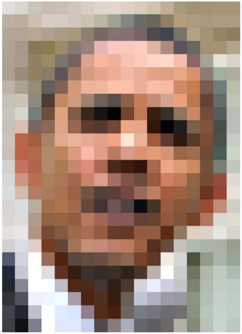
Minimization (convex regularization)

$$\arg\min_{\mathbf{x}} \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2 + \lambda \|\mathbf{x}\|_{\mathcal{A}}$$

Reference : (Tang, Bhaskar, Recht et coll., 2013)

Enhance it ! Toward a 2-D super-resolution

Enhance it ! Toward a 2-D super-resolution



Enhance it ! Toward a 2-D super-resolution

Inverse problem

$$y = \mathbf{A}x$$

Example (Operator)

- $\mathbf{A}$ = subsampling
- $\mathbf{A}$ = blurring
- ...

Inverse problem

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

Example (Operator)

- $\mathbf{A}$ = subsampling
- $\mathbf{A}$ = blurring
- ...

Inverse problem

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

Example (Operator)

- $\mathbf{A}$ = subsampling
- $\mathbf{A}$ = blurring
- ...

Inverse problem

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \epsilon$$

Example (Operator)

- $\mathbf{A}$ = subsampling
- $\mathbf{A}$ = blurring
- ...

Inverse problem

$$\mathbf{y} - \mathbf{A}\mathbf{x} = \epsilon$$

Example (Operator)

- $\mathbf{A}$ = subsampling
- $\mathbf{A}$ = blurring
- ...

Inverse problem

Minimization (data fidelity term)

This is an ill-posed problem :

$$\arg\min_{\mathbf{x}} \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2$$

Example (Operator)

- $\mathbf{A}$ = subsampling
- $\mathbf{A}$ = blurring
- ...

Inverse problem

Minimization (convex regularization)

$$\arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|^2 + \boxed{\lambda R(\mathbf{x})}$$

Exemple (Regularizer)

- $R(\mathbf{x}) = \|\nabla \mathbf{x}\|_2^2$ (Tikhonov, 1963)
- $R(\mathbf{x}) = \|\nabla \mathbf{x}\|_1$ (Rudin et al., 1992)
- $R(\mathbf{x}) = \|\mathbf{x}\|_{\mathcal{A}}$ (Chandrasekaran, 2010)

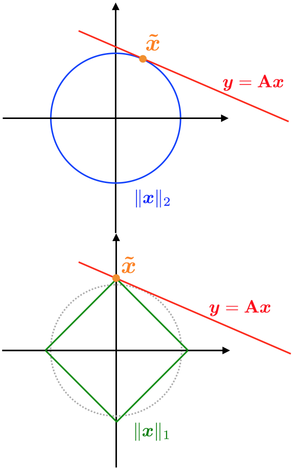
Inverse problem

Minimization (convex regularization)

$$\arg\min_{\mathbf{x}} \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2 + \boxed{\lambda R(\mathbf{x})}$$

Exemple (Regularizer)

- $R(\mathbf{x}) = \|\nabla \mathbf{x}\|_2^2$ (Tikhonov, 1963)
- $R(\mathbf{x}) = \|\nabla \mathbf{x}\|_1$ (Rudin et al., 1992)
- $R(\mathbf{x}) = \|\mathbf{x}\|_{\mathcal{A}}$ (Chandrasekaran, 2010)



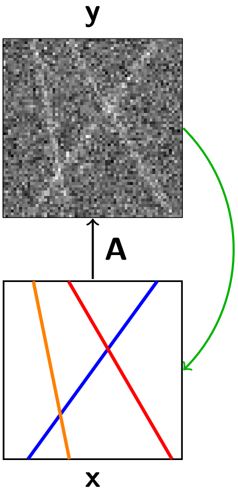
Inverse problem

Minimization (convex regularization)

$$\arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|^2 + \boxed{\lambda R(\mathbf{x})}$$

Exemple (Regularizer)

- $R(\mathbf{x}) = \|\nabla \mathbf{x}\|_2^2$ (Tikhonov, 1963)
- $R(\mathbf{x}) = \|\nabla \mathbf{x}\|_1$ (Rudin et coll., 1992)
- $R(\mathbf{x}) = \|\mathbf{x}\|_{\mathcal{A}}$ (Chandrasekaran, 2010)



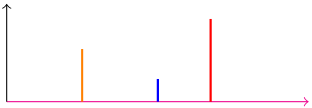
Inverse problem

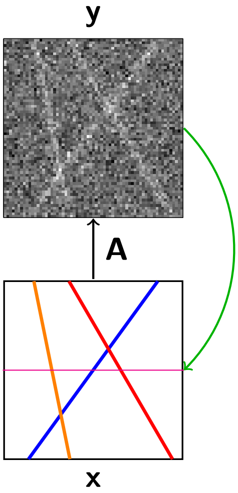
Minimization (convex regularization)

$$\arg\min_{\mathbf{x}} \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2 + \boxed{\lambda R(\mathbf{x})}$$

Exemple (Regularizer)

- $R(\mathbf{x}) = \|\nabla \mathbf{x}\|_2^2$
- $R(\mathbf{x}) = \|\nabla \mathbf{x}\|_1$
- $R(\mathbf{x}) = \|\mathbf{x}\|_{\mathcal{A}}$ parcimonious





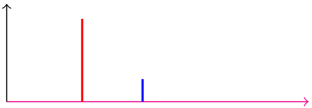
Inverse problem

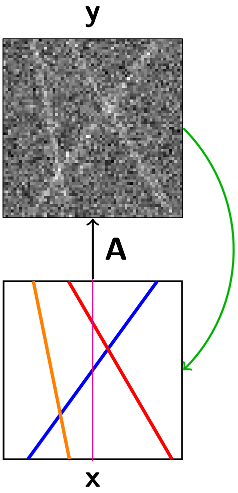
Minimization (convex regularization)

$$\arg\min_{\mathbf{x}} \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2 + \boxed{\lambda R(\mathbf{x})}$$

Exemple (Regularizer)

- $R(\mathbf{x}) = \|\nabla \mathbf{x}\|_2^2$
- $R(\mathbf{x}) = \|\nabla \mathbf{x}\|_1$
- $R(\mathbf{x}) = \|\mathbf{x}\|_{\mathcal{A}}$ parcimonious

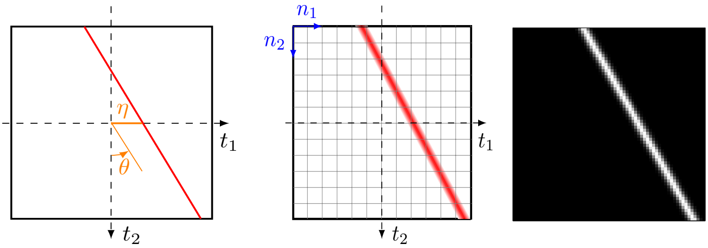




Modeling the perfect lines

$$x^\sharp : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^{K} \alpha_k \delta\left(\cos\theta_k(t_1 - \eta_k) + \sin\theta_k\, t_2\right)$$

$$\mathbf{b}^\sharp[n_1, n_2] = (x^\sharp * \phi)(n_1, n_2), \quad \phi(n_1, n_2) = \boldsymbol{g}[n_1]\boldsymbol{h}[n_2]$$



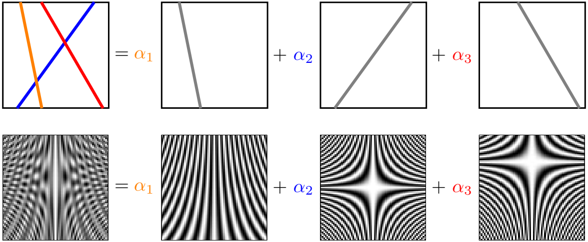
$x^\sharp$ $x^\sharp * \phi$ $\mathbf{b}^\sharp$

Modeling the blurred lines

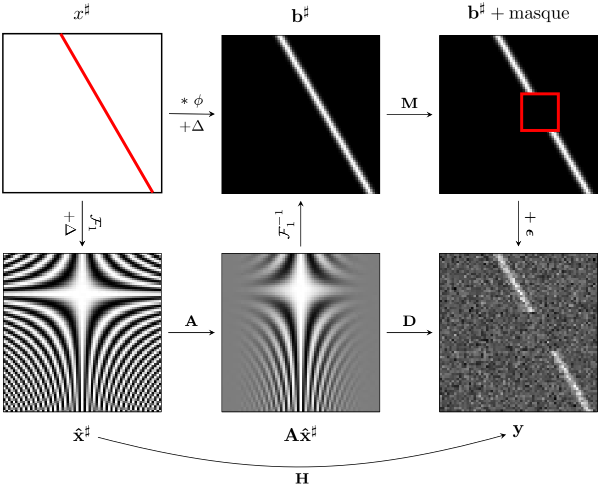
$$\hat{\mathbf{x}}^\sharp[m, n_2] = (\mathcal{F}_1 x^\sharp)[m, n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right) m}$$

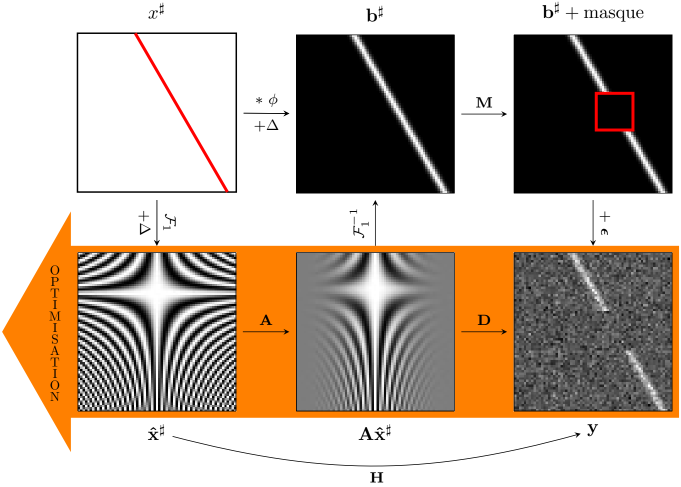
$$c_k = \frac{\alpha_k}{\cos\theta_k} \geq 0$$

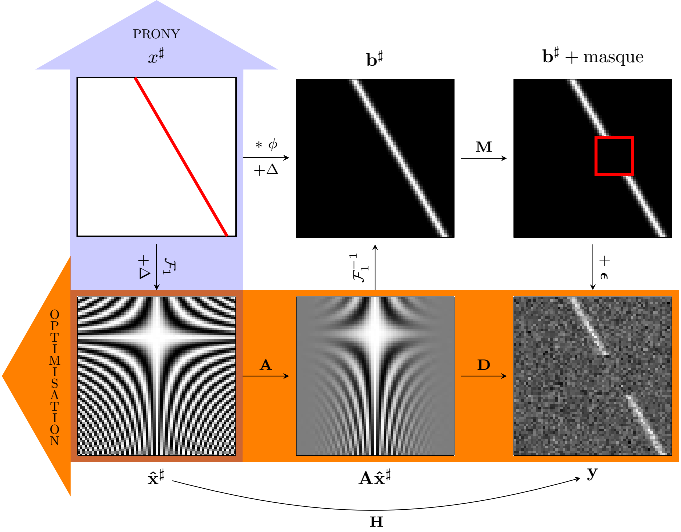
$$\hat{\mathbf{b}}^\sharp[m, :] = (\hat{g}[m]\hat{\mathbf{x}}^\sharp[m, :]) * \boldsymbol{h} \rightarrow \mathbf{A}\hat{\mathbf{x}}^\sharp = \hat{\mathbf{b}}^\sharp$$



Reconstruction steps




$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$* \phi$
$+\Delta$
M
$+\Delta$
$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
$+\epsilon$
OPTIMISATION
A
D
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
y
H


PRONY
$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$*\ \phi$
$+\Delta$
M
$+\Delta$
$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
$+\ \epsilon$
OPTIMISATION
A
D
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
H

Atomic decomposition of the columns

$$\hat{\mathbf{x}}^\sharp[m, n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right)m}$$

$$\boldsymbol{l}_{n_2}^\sharp = \hat{\mathbf{x}}^\sharp[:, n_2] = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k}, 0), \quad [\boldsymbol{a}(f, \phi)]_i = \mathrm{e}^{\mathrm{j}(2\pi f i + \phi)} \in \mathcal{A}$$

Atomic decomposition of the rows

$$\hat{\mathbf{x}}^\sharp[m, n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} m\right) n_2 + \frac{2\pi\eta_k m}{W}}$$

$$\boldsymbol{t}_m^\sharp = \hat{\mathbf{x}}^\sharp[m,:] = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^{\mathsf{T}}, \quad [\boldsymbol{a}(f,\phi)]_i = \mathrm{e}^{\mathrm{j}(2\pi f i + \phi)} \in \mathcal{A}$$

Atomic decomposition of the columns and rows

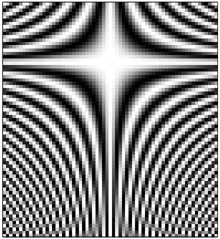
$$\hat{\mathbf{x}}^\sharp[m, n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right) m}$$

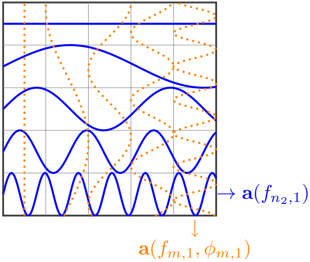
1. $\boldsymbol{l}^\sharp_{n_2} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k}, 0)$ (columns of $\hat{\mathbf{x}}$, without phase)
2. $\boldsymbol{t}^\sharp_{m} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^\mathsf{T}$ (rows of $\hat{\mathbf{x}}$, with phase)

Atomic decomposition of one line ($K = 1$)

$$\hat{\mathbf{x}}^\sharp[m, n_2] = c_1 \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_1}{W} n_2 + \frac{\eta_1}{W}\right)m}$$

1. $\boldsymbol{l}^\sharp_{n_2} = c_1 \boldsymbol{a}(f_{n_2,1}, 0)$ (one atom without phase)
2. $\boldsymbol{t}^\sharp_m = c_1 \boldsymbol{a}(f_{m,1}, \phi_{m,1})^\mathsf{T}$ (one atom with phase)





Atomic norms

$$\hat{\mathbf{x}}^\sharp[m, n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right)m}, \quad c^\star = \sum_{k=1}^{K} c_k$$

1. $\boldsymbol{l}_{n_2}^\sharp = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k}, 0)$ (columns of $\hat{\mathbf{x}}$, without phase)
2. $\boldsymbol{t}_m^\sharp = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^{\mathrm{T}}$ (rows of $\hat{\mathbf{x}}$, with phase)

Atomic norm :

$$\|z\|_{\mathcal{A}} = \inf_{c'_k, f'_k, \phi'_k} \left\{ \sum_k c'_k : z = \sum_k c'_k \boldsymbol{a}(f'_k, \phi'_k) \right\}$$

Atomic norms characterization

1. $\boldsymbol{l}_{n_2}^{\sharp} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k}, 0)$

$\hookrightarrow \mathbf{T}_{M+1}(\boldsymbol{l}_{n_2}^{\sharp}) \succcurlyeq 0$ + of rank K (Carathéodory, 1907)

$\hookrightarrow \|\boldsymbol{l}_{n_2}^{\sharp}\|_{\mathcal{A}} = \sum_{k=1}^{K} c_k = \hat{\mathbf{x}}^{\sharp}[0, n_2]$

2. $\boldsymbol{t}_{m}^{\sharp} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^{\mathsf{T}}$ (Tang et al., 2013)

$$\|\boldsymbol{t}_m^{\sharp}\|_{\mathcal{A}} = \inf_{\boldsymbol{q}\in\mathbb{C}^N, t\in\mathbb{R}} \left\{ \frac{1}{2}\mathrm{Tr}(\mathbf{T}_N(\boldsymbol{q})) + \frac{1}{2}t : \begin{pmatrix} \mathbf{T}_N(\boldsymbol{q}) & \boldsymbol{t}_m^{\sharp} \\ \boldsymbol{t}_m^{\sharp\,*} & t \end{pmatrix} \succcurlyeq 0 \right\} .$$

Atomic norms characterization

1. $\boldsymbol{l}^{\sharp}_{n_2} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k}, 0)$

$\hookrightarrow \mathbf{T}_{M+1}(\boldsymbol{l}^{\sharp}_{n_2}) \succcurlyeq 0$ + of rank K (Carathéodory, 1907)

$\hookrightarrow \|\boldsymbol{l}^{\sharp}_{n_2}\|_{\mathcal{A}} = \sum_{k=1}^{K} c_k = \hat{\mathbf{x}}^{\sharp}[0, n_2] = c^{\star}$

2. $\boldsymbol{t}^{\sharp}_{m} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^{\mathsf{T}}$ (Polisano et al., 2016)

$$\|\boldsymbol{t}^{\sharp}_{m}\|_{\mathcal{A}} = \min_{\boldsymbol{q} \in \mathbb{C}^N} \left\{ q_0 : \underbrace{\begin{pmatrix} \mathbf{T}_N(\boldsymbol{q}) & \boldsymbol{t}^{\sharp}_{m} \\ \boldsymbol{t}^{\sharp\,*}_{m} & q_0 \end{pmatrix}}_{\mathbf{T}'_N(\boldsymbol{t}^{\sharp}_{m}, \boldsymbol{q})} \succcurlyeq 0 \right\} \equiv \mathrm{SDP}(\boldsymbol{t}^{\sharp}_{m}),$$

$\hookrightarrow \|\boldsymbol{t}^{\sharp}_{m}\|_{\mathcal{A}} = \mathrm{SDP}(\boldsymbol{t}^{\sharp}_{m}) = \boldsymbol{q}_m[0] \leqslant c^{\star}$

Atomic norms characterization

$$\hat{\mathbf{x}}^\sharp[m,n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right)m}, \quad c^\star = \sum_{k=1}^{K} c_k$$

1. $\boldsymbol{l}_{n_2}^\sharp = \sum_{k=1}^K c_k \boldsymbol{a}(f_{n_2,k}, 0)$ (columns of $\hat{\mathbf{x}}$, without phase)
2. $\boldsymbol{t}_m^\sharp = \sum_{k=1}^K c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^\mathsf{T}$ (rows de $\hat{\mathbf{x}}$, with phase)

Convex regularization of the K lines by the atomic norm

1. $\|\boldsymbol{l}_{n_2}^\sharp\|_{\mathcal{A}} = c^\star = \hat{\mathbf{x}}^\sharp[0,n_2]$ and $\mathbf{T}_{M+1}(\boldsymbol{l}_{n_2}^\sharp) \succcurlyeq 0$
2. $\|\boldsymbol{t}_m^\sharp\|_{\mathcal{A}} = \mathrm{SDP}(\boldsymbol{t}_m^\sharp) = \boldsymbol{q}_m[0] \leqslant c^\star$, $\mathbf{T}'_{H_S}(\boldsymbol{t}_m^\sharp, \boldsymbol{q}_m) \succcurlyeq 0$

Convex optimization problem

Proposition (Convex minimization)

$$\tilde{\mathbf{x}} \in \underset{\hat{\mathbf{x}},\mathbf{q} \in \mathcal{X} \times \mathcal{Q}}{\arg\min} \frac{1}{2} \|\mathbf{A}\hat{\mathbf{x}} - \hat{\mathbf{y}}\|^2 \ ,$$

$$\text{under constraints} \quad \begin{cases} \forall n_2 = 0, \dots, H_S - 1,\ \forall m = 1, \dots, M \ , \\ \hat{\mathbf{x}}[0, n_2] = \hat{\mathbf{x}}[0, 0] \leqslant c \ , \\ \mathbf{q}[m, 0] \leqslant c \ , \\ \mathbf{T}'_{H_S}(\hat{\mathbf{x}}[m, :], \mathbf{q}[m, :]) \succcurlyeq 0 \ , \\ \mathbf{T}_{M+1}(\hat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 \ . \end{cases}$$

Convex optimization problem

Proposition (Convex minimization)

$$\tilde{\mathbf{x}} \in \underset{\hat{\mathbf{x}},\mathbf{q} \in \mathcal{X}\times\mathcal{Q}}{\arg\min} \frac{1}{2}\|\mathbf{A}\hat{\mathbf{x}} - \hat{\mathbf{y}}\|^2 ,$$

$$\text{under constraints} \quad \begin{cases} \forall n_2 = 0, \dots, H_S - 1,\ \forall m = 1, \dots, M , \\ \hat{\mathbf{x}}[0, n_2] = \hat{\mathbf{x}}[0, 0] \leqslant c , \\ \mathbf{q}[m, 0] \leqslant c , \\ \mathbf{T}'_{H_S}(\hat{\mathbf{x}}[m, :], \mathbf{q}[m, :]) \succcurlyeq 0 , \\ \mathbf{T}_{M+1}(\hat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 . \end{cases}$$

(Chambolle et Pock, 2010)

$$\tilde{\mathbf{X}} = \underset{\mathbf{X}\in\mathcal{H}}{\arg\min} \left\{ F(\mathbf{X}) + G(\mathbf{X}) + \sum_{i=0}^{Q-1} H_i(\mathrm{L}_i(\mathbf{X})) \right\}$$

Convex optimization problem

Proposition (Convex minimization)

$$\tilde{\mathbf{x}} \in \underset{\hat{\mathbf{x}},\mathbf{q} \in \mathcal{X}\times\mathcal{Q}}{\arg\min} \frac{1}{2}\|\mathbf{A}\hat{\mathbf{x}} - \hat{\mathbf{y}}\|^2 ,$$

$$\text{under constraints} \quad \begin{cases} \forall n_2 = 0, \ldots, H_S - 1, \ \forall m = 1, \ldots, M , \\ \hat{\mathbf{x}}[0, n_2] = \hat{\mathbf{x}}[0,0] \leqslant c , \\ \mathbf{q}[m, 0] \leqslant c , \\ \mathbf{T}'_{H_S}(\hat{\mathbf{x}}[m,:], \mathbf{q}[m,:]) \succcurlyeq 0 , \\ \mathbf{T}_{M+1}(\hat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 . \end{cases}$$

(Chambolle et Pock, 2010)

$$\tilde{\mathbf{X}} = \underset{\mathbf{X}\in\mathcal{H}}{\arg\min} \left\{ F(\mathbf{X}) + G(\mathbf{X}) + \sum_{i=0}^{Q-1} H_i(\mathrm{L}_i(\mathbf{X})) \right\}$$

Convex optimization problem

Proposition (Convex minimization)

$$\tilde{\mathbf{x}} \in \underset{\hat{\mathbf{x}},\mathbf{q} \in \mathcal{X}\times\mathcal{Q}}{\arg\min} \frac{1}{2}\|\mathbf{A}\hat{\mathbf{x}} - \hat{\mathbf{y}}\|^2 ,$$

under constraints

$$\begin{cases} \forall n_2 = 0, \dots, H_S - 1,\ \forall m = 1, \dots, M , \\ \hat{\mathbf{x}}[0, n_2] = \hat{\mathbf{x}}[0,0] \leqslant c , \\ \mathbf{q}[m,0] \leqslant c , \\ \mathbf{T}'_{H_S}(\hat{\mathbf{x}}[m,:], \mathbf{q}[m,:]) \succcurlyeq 0 , \\ \mathbf{T}_{M+1}(\hat{\mathbf{x}}[:,n_2]) \succcurlyeq 0 . \end{cases}$$

(Chambolle et Pock, 2010)

$$\tilde{\mathbf{X}} = \underset{\mathbf{X}\in\mathcal{H}}{\arg\min} \left\{ F(\mathbf{X}) + G(\mathbf{X}) + \sum_{i=0}^{Q-1} H_i(\mathrm{L}_i(\mathbf{X})) \right\}$$

Convex optimization problem

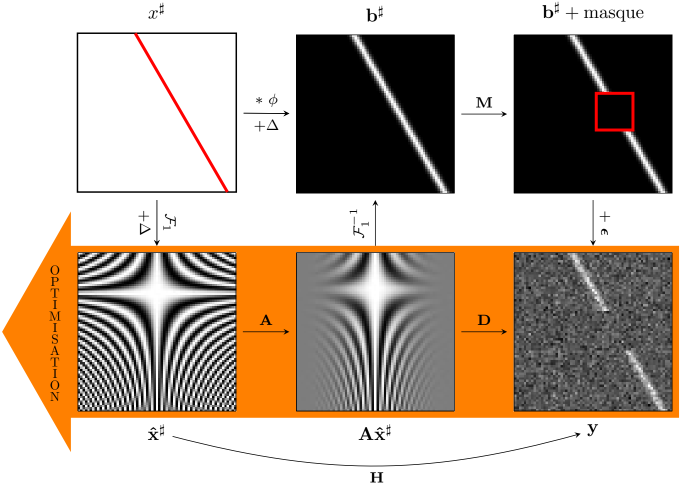
Proposition (Convex minimization)

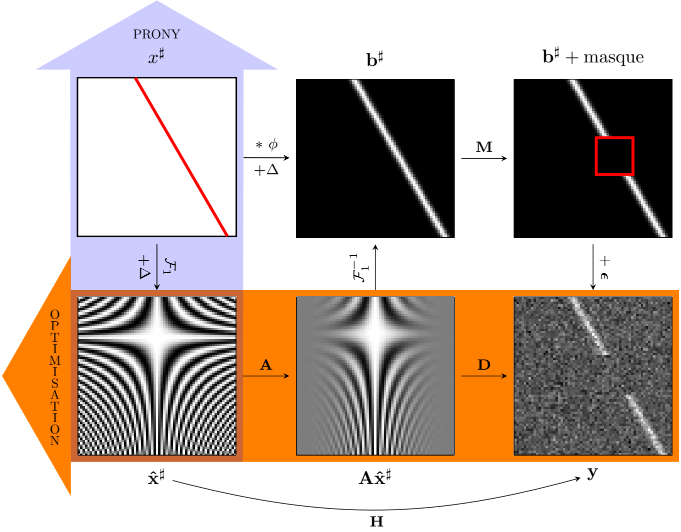
$$\tilde{\mathbf{x}} \in \underset{\hat{\mathbf{x}},\mathbf{q} \in \mathcal{X}\times\mathcal{Q}}{\arg\min} \frac{1}{2}\|\mathbf{A}\hat{\mathbf{x}} - \hat{\mathbf{y}}\|^2 ,$$

$$\text{under constraints} \quad \begin{cases} \forall n_2 = 0, \ldots, H_S - 1,\ \forall m = 1, \ldots, M , \\ \hat{\mathbf{x}}[0, n_2] = \hat{\mathbf{x}}[0,0] \leqslant c , \\ \mathbf{q}[m, 0] \leqslant c , \\ \mathbf{T}'_{H_S}(\hat{\mathbf{x}}[m, :], \mathbf{q}[m, :]) \succcurlyeq 0 , \\ \mathbf{T}_{M+1}(\hat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 . \end{cases}$$

(Chambolle and Pock, 2010)

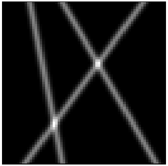
$$\tilde{\mathbf{X}} = \underset{\mathbf{X} \in \mathcal{H}}{\arg\min} \left\{ F(\mathbf{X}) + G(\mathbf{X}) + \sum_{i=0}^{Q-1} H_i(\mathrm{L}_i(\mathbf{X})) \right\}$$


$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$*\ \phi$
$+\Delta$
$\mathbf{M}$
$+\Delta$
$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
$+\epsilon$
OPTIMISATION
$\mathbf{A}$
$\mathbf{D}$
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
$\mathbf{H}$

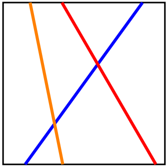

PRONY
$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$* \phi$
$+\Delta$
$\mathbf{M}$
$\mathcal{F}_1$
$+\Delta$
$\mathcal{F}_1^{-1}$
$+\epsilon$
OPTIMISATION
$\mathbf{A}$
$\mathbf{D}$
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
$\mathbf{H}$

Numerical experiments

- Denoising and deconvolution



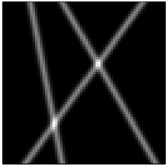
Exp. 1



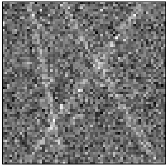
Détection

Numerical experiments

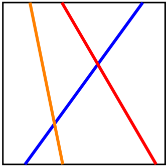
- Denoising and deconvolution



Exp. 1



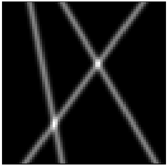
Exp. 2



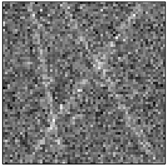
Détection

Numerical experiments

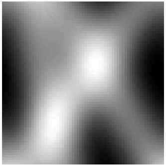
- Denoising and deconvolution



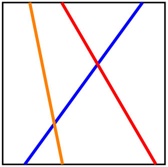
Exp. 1



Exp. 2



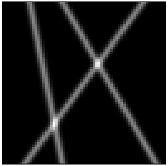
Exp 3.

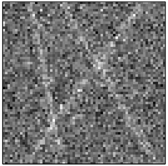


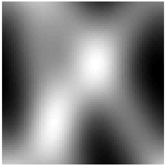
Détection

Numerical experiments

- Denoising and deconvolution


Exp. 1


Exp. 2


Exp 3.

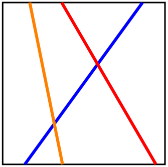
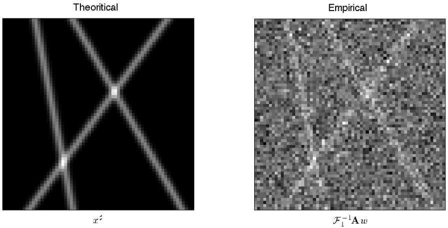

Détection

Table: Relative errors of the line parameters estimation

	Expérience 1	Expérience 2	Expérience 3
Δ_θ/θ	$(10^{-7}, 3.10^{-6}, 7.10^{-7})$	$(10^{-2}, 6.10^{-2}, 9.10^{-2})$	$(6.10^{-7}, 9.10^{-5}, 8.10^{-6})$
Δ_α/α	$(10^{-7}, 10^{-7}, 10^{-7})$	$(10^{-2}, 9.10^{-2}, 2.10^{-1})$	$(4.10^{-5}, 2.10^{-5}, 2.10^{-5})$
Δ_η	$(4.10^{-6}, 7.10^{-6}, 7.10^{-6})$	$(5.10^{-2}, 4.10^{-2}, 3.10^{-2})$	$(5.10^{-5}, 10^{-4}, 3.10^{-4})$

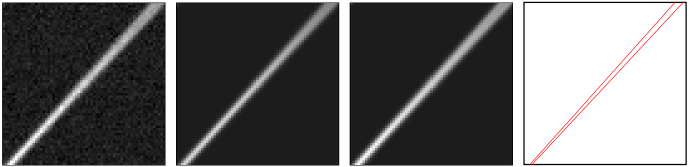
Numerical experiments

image000



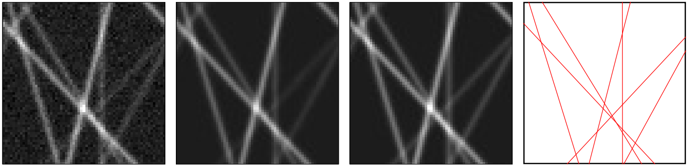
Numerical experiments

- Closed lines



Noisy | Noisy | Ground truth | Detection

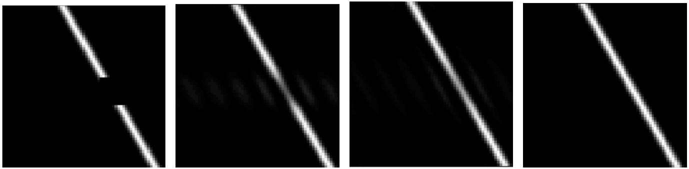
- Multiple lines



Noisy | Noisy | Ground trutht | Detection

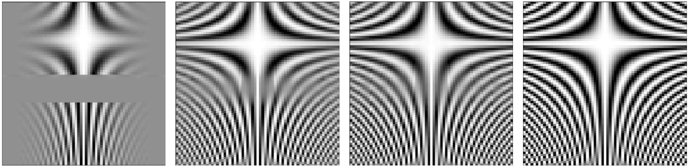
Numerical experiments

- Spatial inpainting



Masking | iter $= 2000$ | iter $= 10000$ | iter $\to \infty$

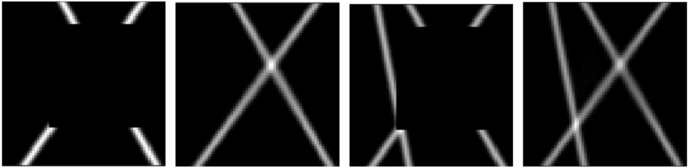
- Inpainting in Fourier



Masquage | iter $= 2000$ | iter $= 10000$ | iter $\to \infty$

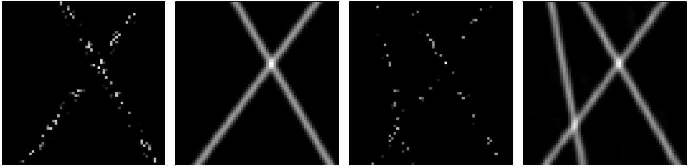
Numerical experiments

- Inpainting with big mask



Masquage Inpainting Masquage Inpainting

- Inpainting with random mask



Masquage Inpainting Masquage Inpainting

Outline

Take home message

- Two new models of anisotropic Gaussian fields producing textures with prescribed local orientation and regularity
- Efficient methods for the simulation of these models
- A local orientation notion for a large class of random fields
- Characterization of the statistic estimators for the orientation and directionality parameters
- New method for the super-resolution of 2-D lines
- Penalize in both directions can lead to the exact solution
- Toward the super-resolution of 2-D curves ?

The tangent field: a tool for analysis and synthesis

1. A tool for analysis (Lévy-Vehel, 1995), (Falconer, 2002) :

$$\left\{ \lim_{\rho \to 0} \frac{X(\mathbf{x_0} + \rho \mathbf{x}) - X(\mathbf{x_0})}{\rho^{h(\mathbf{x}_0)}} \right\}_{\mathbf{x} \in \mathbb{R}^2} \stackrel{d}{=} \{ Y_{\mathbf{x}_0}(\mathbf{x}) \}_{\mathbf{x} \in \mathbb{R}^2}$$

2. A tool for synthesis (Lévy-Véhel, 1995), (Benassi, 1997) :

$$X(\mathbf{x}_0) \leftarrow Y_{\mathbf{x}_0}(\mathbf{x} = \mathbf{x}_0)$$

Multifractional Brownian field B^h (MBF) (Peltier, Vehel, 1995)

- Analysis : the MBF behaves locally as a FBF

$$\left\{ \lim_{\rho \to 0} \frac{B^h(\mathbf{x_0} + \rho \mathbf{x}) - B^h(\mathbf{x_0})}{\rho^{h(\mathbf{x}_0)}} \right\}_{\mathbf{x} \in \mathbb{R}^2} \stackrel{d}{=} \{ B^{h(\mathbf{x}_0)}(\mathbf{x}) \}_{\mathbf{x} \in \mathbb{R}^2}$$

- Synthesis : $B^h(\mathbf{x}_0) \leftarrow B^{h(\mathbf{x}_0)}(\mathbf{x} = \mathbf{x}_0)$

The tangent field: a tool for analysis and synthesis

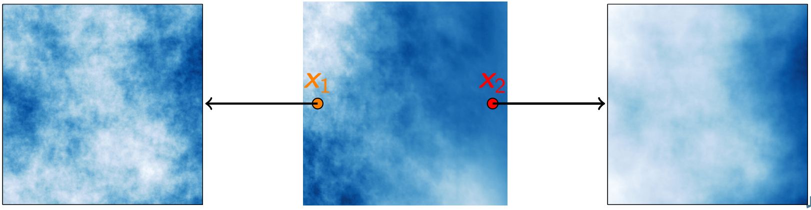
1. A tool for analysis (Lévy-Vehel, 1995), (Falconer, 2002) :

$$\left\{ \lim_{\rho \to 0} \frac{X(\mathbf{x_0} + \rho \mathbf{x}) - X(\mathbf{x_0})}{\rho^{h(\mathbf{x}_0)}} \right\}_{\mathbf{x} \in \mathbb{R}^2} \overset{d}{=} \{ Y_{\mathbf{x}_0}(\mathbf{x}) \}_{\mathbf{x} \in \mathbb{R}^2}$$

2. A tool for synthesis (Lévy-Véhel, 1995), (Benassi, 1997) :

$$X(\mathbf{x}_0) \leftarrow Y_{\mathbf{x}_0}(\mathbf{x} = \mathbf{x}_0)$$

FBF B^H, $H \equiv h(\mathbf{x}_1)$ MBF $B^h(\mathbf{x})$ FBF B^H, $H \equiv h(\mathbf{x}_2)$



The tangent field: a tool for analysis and synthesis

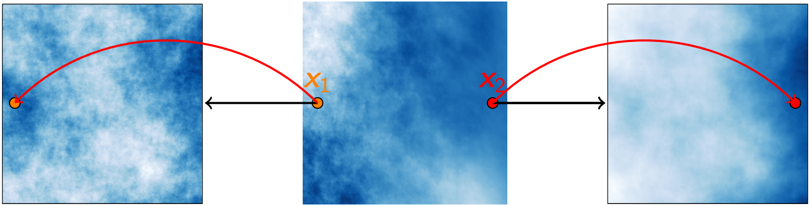
1. A tool for analysis (Lévy-Vehel, 1995), (Falconer, 2002) :

$$\left\{ \lim_{\rho \to 0} \frac{X(\boldsymbol{x_0} + \rho \boldsymbol{x}) - X(\boldsymbol{x_0})}{\rho^{h(\boldsymbol{x}_0)}} \right\}_{\boldsymbol{x} \in \mathbb{R}^2} \stackrel{d}{=} \{ Y_{\boldsymbol{x}_0}(\boldsymbol{x}) \}_{\boldsymbol{x} \in \mathbb{R}^2}$$

2. A tool for synthesis (Lévy-Véhel, 1995), (Benassi, 1997) :

$$X(x_0) \leftarrow Y_{\boldsymbol{x}_0}(\boldsymbol{x} = x_0)$$

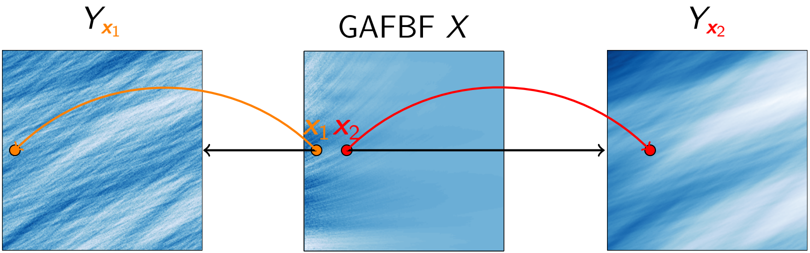
B^H, $H \equiv h(x_1)$ | MBM $B^{h(\boldsymbol{x})}$ | B^H, $H \equiv h(x_2)$



Synthesis of the GAFBF by its tangent fields

$$X(\mathbf{x}_0) \leftarrow Y_{\mathbf{x}_0}(\mathbf{x} = \mathbf{x}_0) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \mathbf{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C_{\mathbf{x}_0}(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\mathbf{x}_0)+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

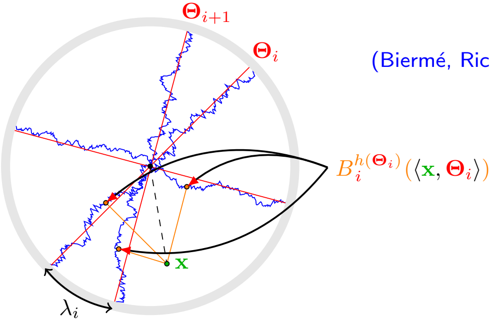
$\Rightarrow$ requires to simulate as many tangent fields there are pixels in the image !



Synthesis of a H-sssi by turning bands

$$Y_{\mathbf{x}_0}^{[n]}(\mathbf{x}) = \sum_{i=1}^{n} \omega_i(\mathbf{x}_0) B_i^H(\langle \mathbf{x}, \Theta_i \rangle) \ ,$$

$$\omega_i(\mathbf{x}_0)^2 = \lambda_i \gamma(h(\mathbf{x}_0)) C_{\mathbf{x}_0}(\Theta_i)$$



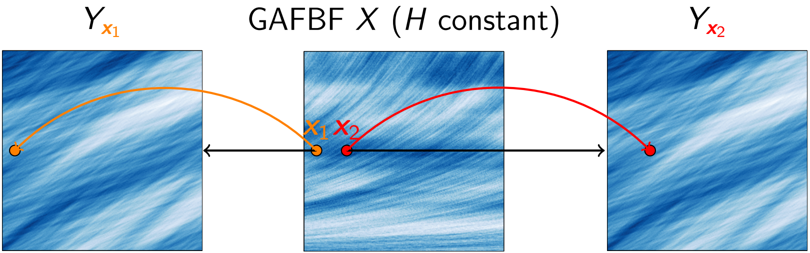
(Matheron, 1973)

(Biermé, Richard, Moisan, 2015)

Synthesis of GAFBF inspired from (Wood, 1994)

1. Simulate U GAFBF X^{H_u} with constant regularities $(H_u)_{1 \leq u \leq U}$:

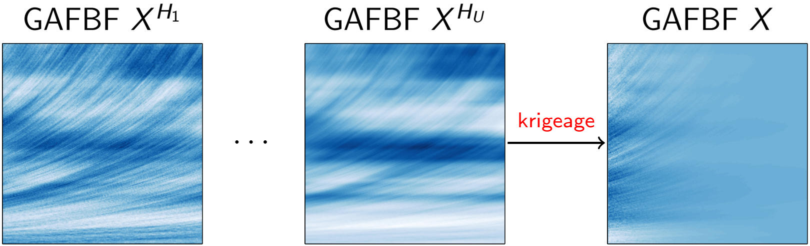
$$X^{H_u}(\mathbf{x}_0) \leftarrow Y_{\mathbf{x}_0}(\mathbf{x} = \mathbf{x}_0) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \mathbf{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C_{\mathbf{x}_0}(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{H_u+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$



Synthesis of GAFBF inspired from (Wood, 1994)

❷ Simulate the GAFBF with variable regularity by krigeage :

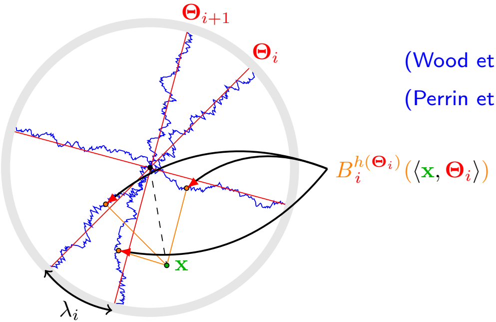
Spatial interpolation of the (X^{H_u}) from the covariance



Synthesis of H-sssi fields by turning bands

$$Y_{\mathbf{x}_0}^{[n]}(\mathbf{x}) = \sum_{i=1}^{n} \omega_i(\mathbf{x}_0) B_i^H(\langle \mathbf{x}, \mathbf{\Theta}_i \rangle) \ ,$$

$\Rightarrow$ Simulate n FBM B_i^H of complexity $O(\ell \log \ell)$



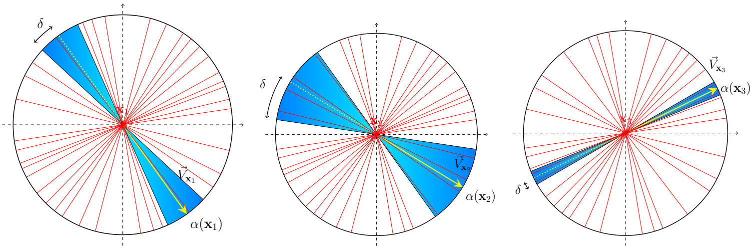
(Wood et coll., 1994)

(Perrin et coll., 2002)

Simulation of the LAFBF with H constant

$$B^H_{\alpha,\delta}(\mathbf{x}_0) \leftarrow Y^{[n]}_{\mathbf{x}_0}(\mathbf{x} = \mathbf{x}_0) = \sum_{i=1}^{n} \omega_i(\mathbf{x}_0) B^H_i(\langle \mathbf{x}_0, \mathbf{\Theta}_i \rangle) \ ,$$

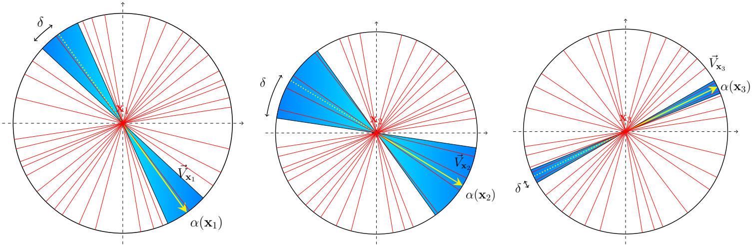
$$\omega_i(\mathbf{x}_0)^2 \propto C_{\mathbf{x}_0}(\mathbf{\Theta}_i) = \mathbb{1}_{[-\delta(\mathbf{x}_0),\delta(\mathbf{x}_0)]}(\arg \mathbf{\Theta}_i - \alpha(\mathbf{x}_0))$$



Simulation of the LAFBF with H constant

$$B^H_{\alpha,\delta}(\boldsymbol{x}_0) \leftarrow Y^{[n]}_{\boldsymbol{x}_0}(\boldsymbol{x} = \boldsymbol{x}_0) = \sum_{i=1}^{n} \omega_i(\boldsymbol{x}_0) B^H_i(\langle \boldsymbol{x}_0, \boldsymbol{\Theta}_i \rangle) \ ,$$

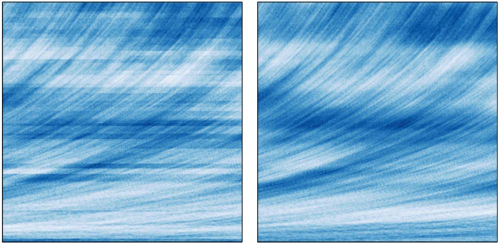
- Pre-computing of the n B^H_i (complexity $O(\ell \log \ell)$)
- The rest of the algorithm is of complexity $O(\log n \# \text{pixels})$



Simulation of the LAFBF with H constant

$$B^H_{\alpha,\delta} \leftarrow Y^{[n]}_{\mathbf{x}_0}(\mathbf{x} = \mathbf{x}_0) = \sum_{i=1}^{n} \omega_i(\mathbf{x}_0) B^H_i(\langle \mathbf{x}_0, \boldsymbol{\Theta}_i \rangle) \ ,$$

$$\omega_i(\mathbf{x}_0)^2 \propto C_{\mathbf{x}_0}(\boldsymbol{\Theta}_i) = \mathbb{1}_{[-\delta(\mathbf{x}_0),\delta(\mathbf{x}_0)]}(\arg \boldsymbol{\Theta}_i - \alpha(\mathbf{x}_0))$$

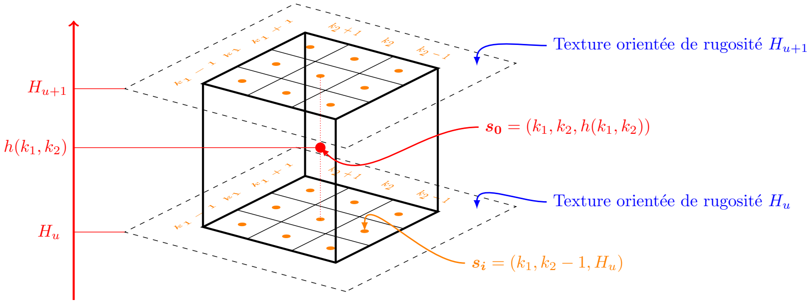


$C_{\mathbf{x}_0}(\boldsymbol{\Theta}_i)$ $\widetilde{C}_{\mathbf{x}_0}(\boldsymbol{\Theta}_i)$ régularisée

Simulation of the LAFBF with h variable (krigeage)

$$\widehat{Z}(\boldsymbol{s_0}) = \sum_{i \in \mathcal{V}(\boldsymbol{s_0})} \lambda_i Z(\boldsymbol{s_i}) = \boldsymbol{\lambda}^\mathsf{T} \boldsymbol{Z} \quad \text{(BLUE)}$$

$$Z = B^h_{\alpha,\delta},\ (B^{H_u}_{\alpha,\delta})_{1 \leq u \leq U} \to Z(\boldsymbol{s_i}),\ \boldsymbol{\Sigma}_{ij} = \mathrm{Cov}(Z(\boldsymbol{s_i}), Z(\boldsymbol{s_j})) \to \boldsymbol{\lambda}$$



Local orientation of a deterministic function

Gradient operator

The gradient operator $\nabla : f \mapsto (\partial_{x_1} f, \partial_{x_2} f)$, with the notation $\partial_{x_1} f : \boldsymbol{x} = (x_1, x_2) \mapsto \frac{\partial f}{\partial x_1}(\boldsymbol{x})$, is defined in Fourier domain by:

$$\widehat{\partial_{x_1} f}(\boldsymbol{\omega}) = -\mathrm{j}\omega_1 \widehat{f}(\boldsymbol{\omega}), \quad \widehat{\partial_{x_2} f}(\boldsymbol{\omega}) = -\mathrm{j}\omega_2 \widehat{f}(\boldsymbol{\omega})$$

$\Rightarrow$ Orientation : $\boldsymbol{n}(\boldsymbol{x}) = \frac{\nabla f(\boldsymbol{x})}{\|\nabla f(\boldsymbol{x})\|}$, $\theta(\boldsymbol{x}) = \arctan\left(\frac{\partial_{x_2} f(\boldsymbol{x})}{\partial_{x_1} f(\boldsymbol{x})}\right)$

$\Rightarrow$ (More robust) minimize the directions against ∇f :

$$\max_{\theta'} \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \left\langle \boldsymbol{n}(\theta'), \nabla f(\boldsymbol{x}') \right\rangle^2 \mathrm{d}\boldsymbol{x}' = \max_{\theta'} \boldsymbol{n}(\theta')^\mathsf{T} \mathbf{J}_f^W(\boldsymbol{x}) \boldsymbol{n}(\theta')$$

$$[\mathbf{J}_f^W(\boldsymbol{x})]_{pq} = \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \partial_{x_p} f(\boldsymbol{x}') \partial_{x_q} f(\boldsymbol{x}') \,\mathrm{d}\boldsymbol{x}', \quad p, q \in \{1, 2\}$$

Local orientation of a deterministic function

Riesz transform and monogenic signal (Felsberg, 2001)

The Riesz operator $\boldsymbol{\mathcal{R}} : f \mapsto (\mathcal{R}_1 f, \mathcal{R}_2 f)$ is defined by:

$$\widehat{\mathcal{R}_1 f}(\boldsymbol{\omega}) = -\mathrm{j}\frac{\omega_1}{\|\boldsymbol{\omega}\|}\widehat{f}(\boldsymbol{\omega}), \quad \widehat{\mathcal{R}_2 f}(\boldsymbol{\omega}) = -\mathrm{j}\frac{\omega_2}{\|\boldsymbol{\omega}\|}\widehat{f}(\boldsymbol{\omega})$$

$\Rightarrow$ Orientation : $\boldsymbol{n}(\boldsymbol{x}) = \frac{\boldsymbol{\mathcal{R}}f(\boldsymbol{x})}{\|\boldsymbol{\mathcal{R}}f(\boldsymbol{x})\|}$, $\theta(\boldsymbol{x}) = \arctan\left(\frac{\mathcal{R}_2 f(\boldsymbol{x})}{\mathcal{R}_1 f(\boldsymbol{x})}\right)$

$\Rightarrow$ (More robust) minimize the directions against $\boldsymbol{\mathcal{R}}f$:

$$\max_{\theta'} \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \left\langle \boldsymbol{n}(\theta'), \boldsymbol{\mathcal{R}}f(\boldsymbol{x}')\right\rangle^2 \mathrm{d}\boldsymbol{x}' = \max_{\theta'} \boldsymbol{n}(\theta')^{\mathsf{T}} \mathbf{J}_f^W(\boldsymbol{x}) \boldsymbol{n}(\theta')$$

$$[\mathbf{J}_f^W(\boldsymbol{x})]_{pq} = \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \mathcal{R}_p f(\boldsymbol{x}') \mathcal{R}_q f(\boldsymbol{x}') \,\mathrm{d}\boldsymbol{x}', \quad p, q \in \{1, 2\}$$

Local orientation of a deterministic function

Monogenic wavelet coefficients (Unser, Olhede, 2009)

Let $\psi_{i,\mathbf{k}}(x) = 2^i \psi(2^i x - \mathbf{k})$ be a wavelet frame constructed from a real isotropic wavelet $\widehat{\psi}(\boldsymbol{\xi}) = \varphi(\|\boldsymbol{\xi}\|)$. We consider the wavelet coefficients of $\boldsymbol{\mathcal{R}}f$ in the frame $\{\psi_{i,\mathbf{k}}\}$:

$$c_{i,\mathbf{k}}^{(\mathcal{R})}(f) = \begin{pmatrix} c_{i,\mathbf{k}}^{(1)}(f) \\ c_{i,\mathbf{k}}^{(2)}(f) \end{pmatrix} = \begin{pmatrix} \langle \mathcal{R}_1 f, \psi_{i,\mathbf{k}} \rangle \\ \langle \mathcal{R}_2 f, \psi_{i,\mathbf{k}} \rangle \end{pmatrix} = \begin{pmatrix} \langle f, \mathcal{R}_1 \psi_{i,\mathbf{k}} \rangle \\ \langle f, \mathcal{R}_2 \psi_{i,\mathbf{k}} \rangle \end{pmatrix}$$

Tensor structure of the wavelet coefficients :

$$\mathbf{J}_{f,i}^{W}[\mathbf{k}] = c_{i,\mathbf{k}}^{(\mathcal{R})}(f) c_{i,\mathbf{k}}^{(\mathcal{R})}(f)^* = \begin{pmatrix} |c_{i,\mathbf{k}}^{(1)}(f)|^2 & c_{i,\mathbf{k}}^{(1)}(f) \cdot \overline{c_{i,\mathbf{k}}^{(2)}(f)} \\ \overline{c_{i,\mathbf{k}}^{(1)}}(f) \cdot c_{i,\mathbf{k}}^{(2)}(f) & |c_{i,\mathbf{k}}^{(1)}(f)|^2 \end{pmatrix}$$

The WAFBF : warped H-sssi fields

Definition (WAFBF)

Let X be a H-sssi field and $\boldsymbol{\Phi} : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function. The *Warped Anisotropic Fractional Brownian Field* (WAFBF) $Z_{\boldsymbol{\Phi},X}$ is defined as the deformation of the elementary field X by the application $\boldsymbol{\Phi}$:

$$Z_{\boldsymbol{\Phi},X}(\boldsymbol{x}) = X(\boldsymbol{\Phi}(\boldsymbol{x})) .$$

References about deformations of stationary random fields:

- (Perrin and Senoussi, 1999, 2000)
- (Guyon and Perrin, 2000)

The WAFBF : warped H-sssi fields

Definition (WAFBF)

Let X be a H-sssi field and $\boldsymbol{\Phi} : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function. The WAFBF $Z_{\boldsymbol{\Phi},X}$ is defined as the deformation of the elementary field X by the application $\boldsymbol{\Phi}$:

$$Z_{\boldsymbol{\Phi},X}(\boldsymbol{x}) = X(\boldsymbol{\Phi}(\boldsymbol{x})) \ .$$

Theorem (Tangent field of the WAFBF)

$Z_{\boldsymbol{\Phi},X}$ admits at every point $\boldsymbol{x_0} \in \mathbb{R}^2$ the tangent field:

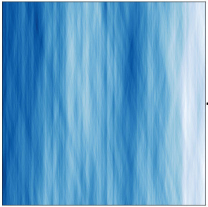
$$Y_{\boldsymbol{x_0}}(\boldsymbol{x}) = X(\mathbf{D}\boldsymbol{\Phi}(\boldsymbol{x_0})\ \boldsymbol{x}) \ , \quad \forall \boldsymbol{x} \in \mathbb{R}^2 \ ,$$

where $\mathbf{D}\boldsymbol{\Phi}(\boldsymbol{x_0})$ is the jacobian matrix of $\boldsymbol{\Phi}$ at point $\boldsymbol{x_0}$.

WAFBF with prescribed local orientations

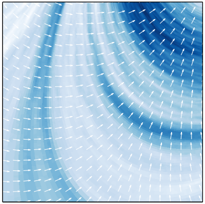
$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$

WAFBF $(a, b) = (2, -1)$



Φ_α

$\alpha(x_1, x_2) = ax_1 + bx_2 + c$



$X_{0,\delta}$

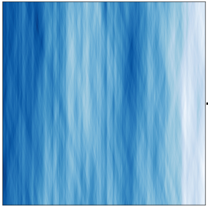
$Z = X_{0,\delta} \circ \Phi_\alpha$

$$\Phi_\alpha(x_1, x_2) = \frac{\exp(ax_2 - bx_1)}{a^2 + b^2} \begin{pmatrix} a\sin(ax_1 + bx_2 + c) - b\cos(ax_1 + bx_2 + c) \\ a\cos(ax_1 + bx_2 + c) + b\sin(ax_1 + bx_2 + c) \end{pmatrix}$$

$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x}) = \frac{\mathbf{D\Phi}(\boldsymbol{x})^{\mathsf{T}}(1,0)}{\|\mathbf{D\Phi}(\boldsymbol{x})^{\mathsf{T}}(1,0)\|} = (\cos\alpha(\boldsymbol{x}), \sin\alpha(\boldsymbol{x}))$$

Warped elementary field

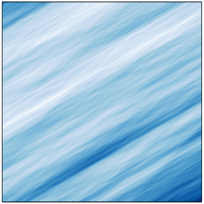
$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$



X

$\xrightarrow[\mathbf{R}_{-\alpha(\mathbf{x})}\mathbf{x}]{\boldsymbol{\Phi}(\mathbf{x})}$

WAFBF

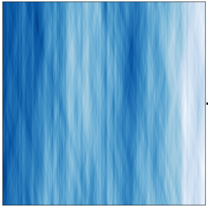


$Z = X \circ \boldsymbol{\Phi}$

$$\alpha(x_1, x_2) = -\frac{\pi}{4}$$

Warped elementary field

$$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$$

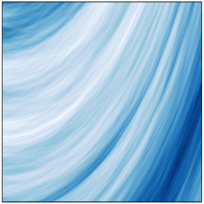


X

$\mathbf{\Phi}(\mathbf{x})$

$\mathbf{R}_{-\alpha(\mathbf{x})}\mathbf{x}$

WAFBF

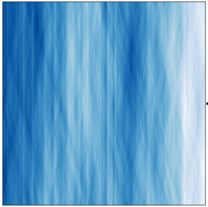


$Z = X \circ \mathbf{\Phi}$

$$\alpha(x_1, x_2) = -\frac{\pi}{2} + x_1$$

Warped elementary field

$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$

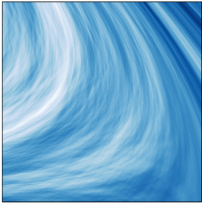


X

$\mathbf{\Phi}(x)$

$\mathbf{R}_{-\alpha(x)} x$

WAFBF

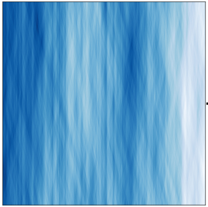


$Z = X \circ \mathbf{\Phi}$

$$\alpha(x_1, x_2) = -\frac{\pi}{2} + x_2$$

Warped elementary field

$C(\boldsymbol{\xi}) = \mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi})$



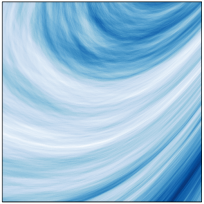
X

$\boldsymbol{\Phi}(\boldsymbol{x})$

$\longrightarrow$

$\mathbf{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}$

WAFBF

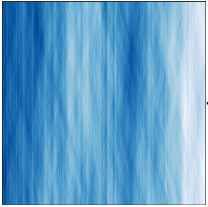


$Z = X \circ \boldsymbol{\Phi}$

$$\alpha(x_1, x_2) = -\frac{\pi}{2} + x_1^2 - x_2$$

Warped elementary field

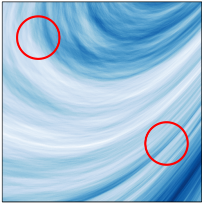
$$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$$



X

$$\xrightarrow[\mathbf{R}_{-\alpha(\mathbf{x})}\mathbf{x}]{\boldsymbol{\Phi}(\mathbf{x})}$$

WAFBF

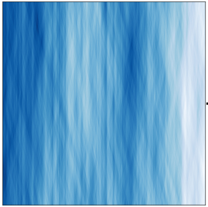


$Z = X \circ \boldsymbol{\Phi}$

1. The directionnality is not controlled

Warped elementary field

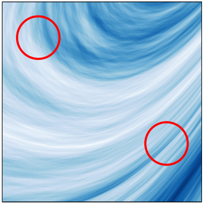
$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$



X

$\xrightarrow[\mathbf{R}_{-\alpha(\mathbf{x})}\mathbf{x}]{\mathbf{\Phi}(\mathbf{x})}$

WAFBF

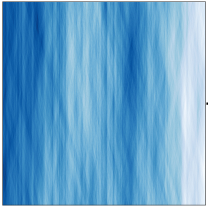


$Z = X \circ \mathbf{\Phi}$

1. The directionnality is not controlled
2. Which transformation $\mathbf{\Phi}$ enables to prescribe the orientation at each point $\alpha(\mathbf{x})$?

Warped elementary field

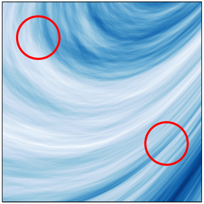
$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$



X

$$\xrightarrow[\mathbf{R}_{-\alpha(x)}x]{\mathbf{\Phi}(x)}$$

WAFBF



$Z = X \circ \mathbf{\Phi}$

1. The directionnality is not controlled
2. Which transformation $\mathbf{\Phi}$ enables to prescribe the orientation at each point $\alpha(x)$?
3. Which definition for the orientation of a random field ?

Orientation of a localizable Gaussian field

Local orientation of the WAFBF where $X = X_{\alpha_0,\delta}$ is an EF

The tangent field of $Z_{\mathbf{\Phi},X}(\boldsymbol{x}) = X_{\alpha_0,\delta}(\mathbf{\Phi}(\boldsymbol{x}))$ at $\boldsymbol{x}_0$ is

$$Y_{\boldsymbol{x}_0}(\boldsymbol{x}) = X_{\alpha_0,\delta}(\mathbf{D\Phi}(\boldsymbol{x}_0)\ \boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^2$$

whose orientation is $\vec{\boldsymbol{n}}_{Y_{\boldsymbol{x}_0}} = \frac{\mathbf{L}^\top \boldsymbol{u}(\alpha_0)}{\|\mathbf{L}^\top \boldsymbol{u}(\alpha_0)\|}$ with $\mathbf{L} = \mathbf{D\Phi}(\boldsymbol{x}_0)$, hence

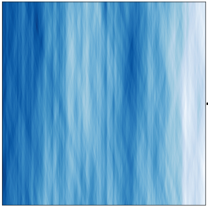
$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x}_0) \equiv \vec{\boldsymbol{n}}_{Y_{\boldsymbol{x}_0}} = \frac{\mathbf{D\Phi}(\boldsymbol{x}_0)^\top \boldsymbol{u}(\alpha_0)}{\|\mathbf{D\Phi}(\boldsymbol{x}_0)^\top \boldsymbol{u}(\alpha_0)\|}$$

Orientation of a localizable Gaussian field

Exemple (Rotation locale du WAFBF où $X = X_{0,\delta}$)

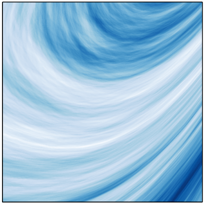
The local orientation of $Z_{\mathbf{\Phi},X}(\mathbf{x}) = X_{0,\delta}(\mathbf{\Phi}(\mathbf{x}))$ at $\mathbf{x}_0$ with $\mathbf{\Phi}(\mathbf{x}) = \mathbf{R}_{-\alpha(\mathbf{x})}\mathbf{x}$ is given by $\vec{\mathbf{n}}_Z(\mathbf{x_0}) = \frac{\mathbf{D\Phi}(\mathbf{x_0})^\top \mathbf{e}_1}{\|\mathbf{D\Phi}(\mathbf{x_0})^\top \mathbf{e}_1\|}$, that is :

$$\vec{\mathbf{n}}_Z(\mathbf{x_0}) = \frac{\mathbf{u}(\alpha(\mathbf{x_0})) + \langle \mathbf{u}(\alpha(\mathbf{x_0}))^\perp, \mathbf{x_0}\rangle \nabla\alpha(\mathbf{x_0})}{\|\mathbf{u}(\alpha(\mathbf{x_0})) + \langle \mathbf{u}(\alpha(\mathbf{x_0}))^\perp, \mathbf{x_0}\rangle \nabla\alpha(\mathbf{x_0})\|}$$



$X_{0,\delta}$

$\mathbf{\Phi}(\mathbf{x})$ → $\mathbf{R}_{-\alpha(\mathbf{x})}\mathbf{x}$



$Z_{\mathbf{\Phi},X} = X_{0,\delta} \circ \mathbf{\Phi}$

Prescribed orientations for the WAFBF

Proposition (Orientation control by harmonic functions)

Let $Z_{\mathbf{\Phi}_\alpha, X}(\boldsymbol{x})$ be the field $X = X_{0,\delta}$ with orientation $\boldsymbol{e}_1 = (1,0)^\mathsf{T}$ warped by a conform transformation $\mathbf{\Phi}_\alpha$ defined by :

1. $\alpha : \mathbb{R}^2 \to \mathbb{R}$ a harmonic function,
2. λ its conjugate harmonic function such as $\mathbf{\Psi}_\alpha = \begin{pmatrix} \lambda \\ -\alpha \end{pmatrix}$ is holomorphic,
3. $\mathbf{\Phi}_\alpha$ a complex primitive of $\exp(\mathbf{\Psi}_\alpha)$.

The local orientation (up to δ^2) of $Z_{\mathbf{\Phi}_\alpha, X}$ at $\boldsymbol{x_0}$ is

$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x_0}) = \begin{pmatrix} \cos \alpha(\boldsymbol{x_0}) \\ \sin \alpha(\boldsymbol{x_0}) \end{pmatrix} = \boldsymbol{u}(\alpha(\boldsymbol{x}_0))$$

Paradigm of the atomic decomposition

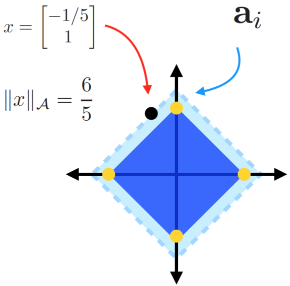
$$\mathbf{x} = \sum_{i=1}^{K} c_i \mathbf{a}_i, \quad c_i \geq 0, \quad \mathbf{a}_i \in \mathcal{A}$$

Atomic norm

$$\|\mathbf{x}\|_{\mathcal{A}} = \inf\left\{t > 0 : \mathbf{x} \in t\,\mathrm{conv}(\mathcal{A})\right\}$$

$$= \inf\left\{\sum_{\mathbf{a}\in\mathcal{A}} c_{\mathbf{a}} : \mathbf{x} = \sum_{\mathbf{a}\in\mathcal{A}} c_{\mathbf{a}}\mathbf{a}\right\}$$

(Chandrasekaran et al., 2010)

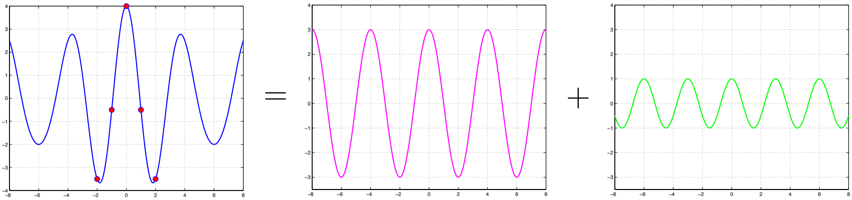


$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$$

$$\|\mathbf{x}\|_{\mathcal{A}} = \|\mathbf{x}\|_1$$

Signals separation

Objective : Extract frequencies and amplitudes of sinusoids.



$$x(t) = x_1(t) + x_2(t) \qquad x_1(t) = \boxed{3}\exp\left(j2\pi\boxed{\frac{1}{4}}t\right) \qquad x_2(t) = \boxed{1}\exp\left(j2\pi\boxed{\frac{1}{3}}t\right)$$

$\Rightarrow$ spectral method estimation (Prony, ESPRIT, MUSIC, ...)

Prony method

$$x_m = \sum_{k=1}^{K} \rho_k \underbrace{\left(e^{-j\omega_k}\right)}_{z_k}^{m}, \quad \rho_k \in \mathbb{C},\ \omega_k \in [-\pi, \pi],\ m = -M, \ldots, M$$

Annihilating filter : $H(z) = \prod_{k=1}^{K} (z - \overline{z_k}) = \sum_{k=0}^{K} h_k z^k$

$$\sum_{j=0}^{K} h_j x_{m-j} = \sum_{j=0}^{K} h_j \left(\sum_{k=1}^{K} \rho_k z_k^{m-j} \right) = \sum_{k=1}^{K} \rho_k z_k^m \underbrace{\left(\sum_{j=0}^{K} h_j z_k^{-j} \right)}_{H(\overline{z_k})=0} = 0$$

Prony method : annihilating polynomial

$$x_m = \sum_{k=1}^{K} \rho_k \underbrace{\left(e^{-j\omega_k}\right)}_{z_k}^{m}, \quad \rho_k \in \mathbb{C},\ \omega_k \in [-\pi, \pi],\ m = -M, \ldots, M$$

Annihilating filter : $H(z) = \prod_{k=1}^{K} (z - \overline{z_k}) = \sum_{k=0}^{K} h_k z^k$

- $\sum_{j=0}^{K} h_j x_{m-j} = 0, \forall m = -M+K, \ldots, M \Leftrightarrow \boldsymbol{x} * \boldsymbol{h} = \boldsymbol{0}$
- $\begin{pmatrix} x_{-M+K} & \cdots & x_{-M} \\ \vdots & \ddots & \vdots \\ x_M & \cdots & x_{M-K} \end{pmatrix} \begin{pmatrix} h_0 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Leftrightarrow \mathbf{T}_K \boldsymbol{h} = \boldsymbol{0}$

Prony method : frequencies estimation

$$x_m = \sum_{k=1}^{K} \rho_k \underbrace{\left(e^{-j\omega_k}\right)}_{z_k}{}^m, \quad \rho_k \in \mathbb{C},\ \omega_k \in [-\pi, \pi],\ m = -M, \dots, M$$

Annihilating filter: $H(z) = \prod_{k=1}^{K} (z - \overline{z_k}) = \sum_{k=0}^{K} h_k z^k$

- $\boldsymbol{h}$ = sing. vec. for $\lambda = 0$ of

$$\mathbf{T}_K = \begin{pmatrix} x_{-M+K} & \cdots & x_{-M} \\ \vdots & \ddots & \vdots \\ x_M & \cdots & x_{M-K} \end{pmatrix}$$

- $\overline{z_k}$ = roots of the polynomial $H(z)$, puis $\omega_k = \arg(\overline{z_k})$

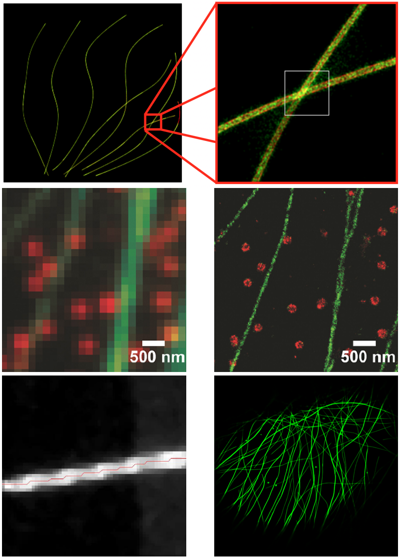
Prony method : amplitudes estimation

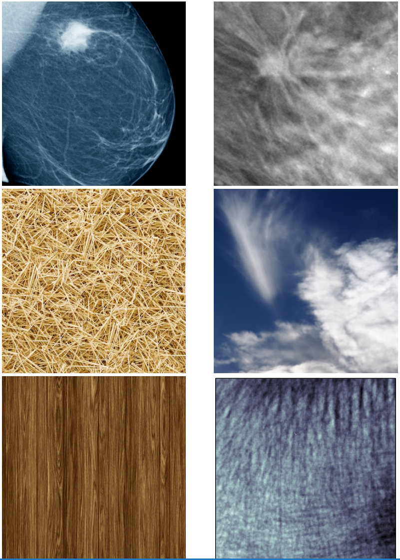
- $x_m = \sum_{k=1}^{K} \rho_k \left(e^{-j\omega_k}\right)^m, \forall m = -M, \dots, M$
- $\begin{pmatrix} e^{jM\omega_1} & \cdots & e^{jM\omega_K} \\ \vdots & \ddots & \vdots \\ e^{-jM\omega_1} & \cdots & e^{-jM\omega_K} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_K \end{pmatrix} = \begin{pmatrix} x_{-M} \\ \vdots \\ x_M \end{pmatrix} \Leftrightarrow \mathbf{U}\boldsymbol{\rho} = \boldsymbol{x}$

Least-square method :

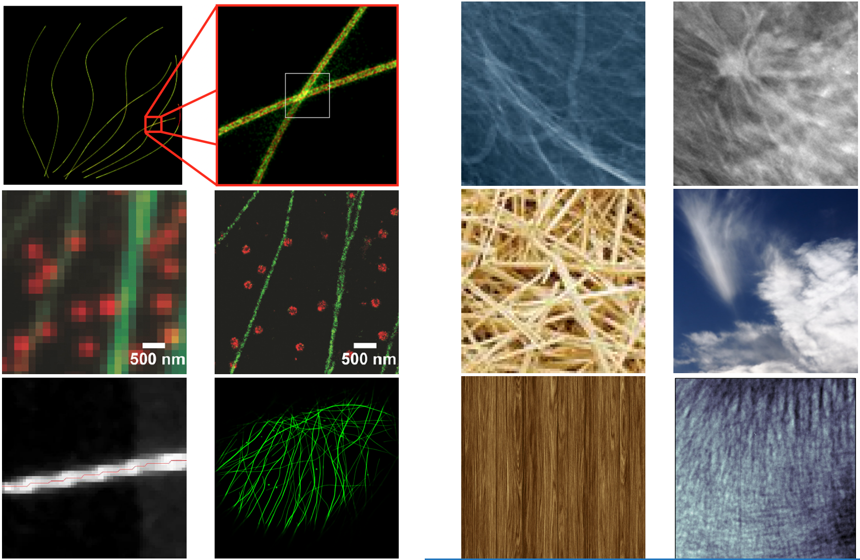
$$\mathbf{U}^{\mathrm{H}}\mathbf{U}\boldsymbol{\rho} = \mathbf{U}^{\mathrm{H}}\boldsymbol{x} \Longleftrightarrow \boldsymbol{\rho} = (\mathbf{U}^{\mathrm{H}}\mathbf{U})^{-1}\mathbf{U}^{\mathrm{H}}\boldsymbol{x}$$

Motivations

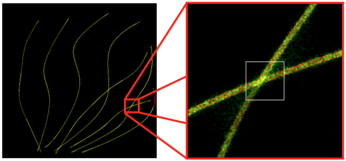


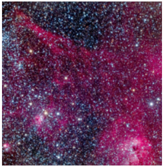


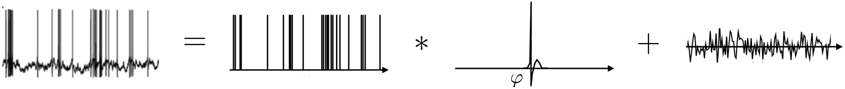
Motivations



Diverses applications







Perpectives

- Improvement of the methods :
  - Definition of the Riesz transform a random field
  - Test hypothesis for the directionality of a texture
  - 2-D extraction of the line parameters
- Applications :
  - Tests of orientation on real medical images
  - Super-resolution of *patchs* on images from microscopy
- Further perspectives :
  - Treat the multiple orientations case
  - Super-resolution of 2-D curves