Riesz-based orientation of localizable Gaussian fields

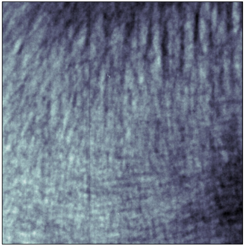
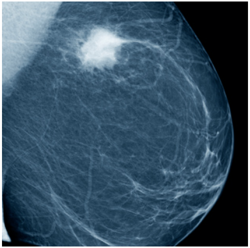
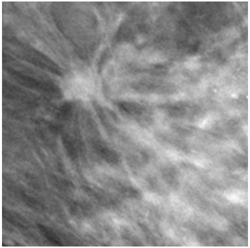
Kévin Polisano

joint work with M. Clausel, L. Condat and V. Perrier

June 4 2021

Motivation: modelling/analysis anisotropic textures




Outline

1. Intro: from Brownian motion to anisotropic random fields

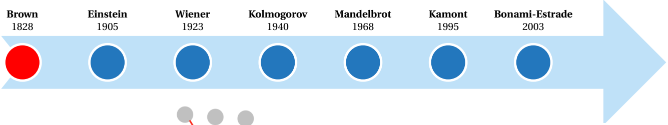
2. Two classes of Gaussian fields with prescribed orientation:
   - Generalized Anisotropic Fractional Brownian Fields (GAFBF)
   - Warped Anisotropic Fractional Brownian Field (WAFBF)

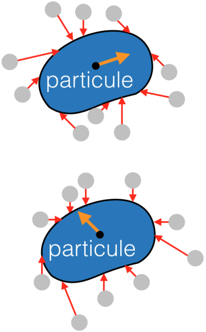
3. Definition of the notion of orientation for random fields:
   - H-self-similar Gaussian fields with stationary increments (H-sssi)
   - Generalization to the class of localizable Gaussian fields

4. Conclusion and perspectives

Introduction: from Brownian motion to anisotropic random fields

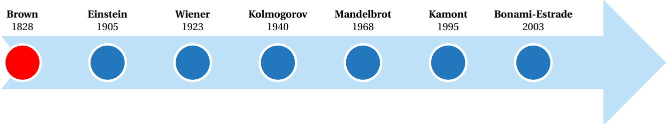
From Brownian to random anisotropic fields

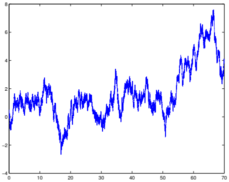


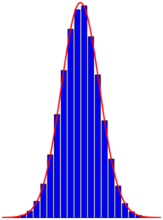


- independants displacements
- Gaussian distribution
- irregular trajectories

From Brownian to random anisotropic fields







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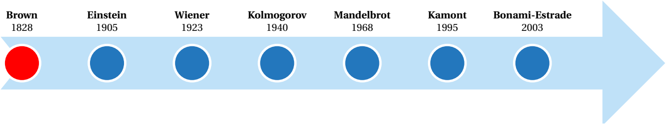

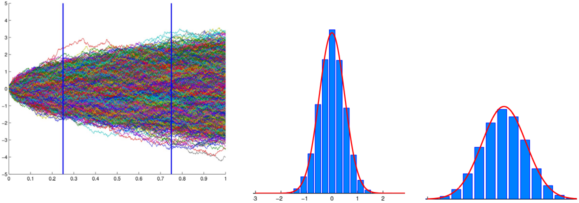

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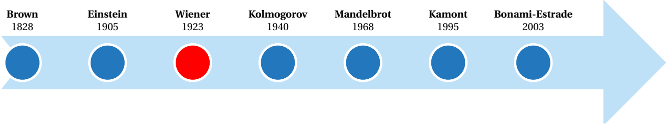
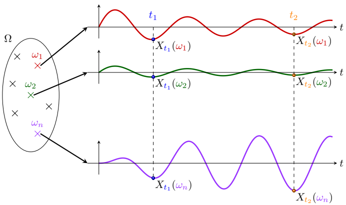


Brownian motion

- $(B_t)_t$ has independants increments, $B_0 = 0$ a.s.
- $B_{t_i} - B_{t_j} \sim \mathcal{N}(0, t_i - t_j)$
- $(B_t)_t$ has continuous sample paths a.s.

$$\begin{array}{cccc} X : & T \times \Omega & \longrightarrow & E \\ & (t, \omega) & \longmapsto & X(t, \omega) \end{array}$$



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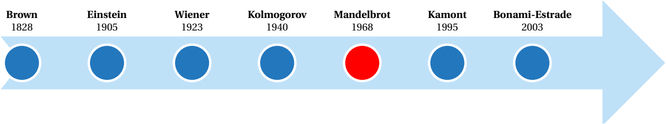
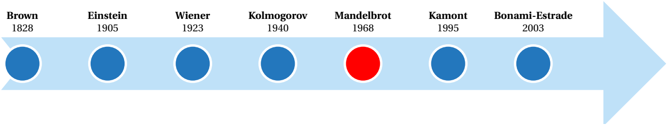


Self-similarity

$\{X(t)\}_{t\in T}$ is self-similar of order H if $\forall\lambda \in \mathbb{R}$

$$\{X(\lambda t)\}_{t\in T} \overset{(fdd)}{=} \lambda^H \{X(t)\}_{t\in T}$$

From Brownian to random anisotropic fields



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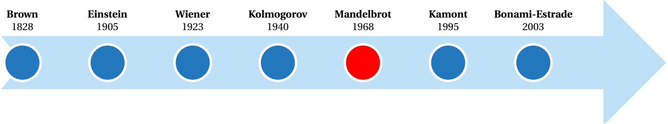

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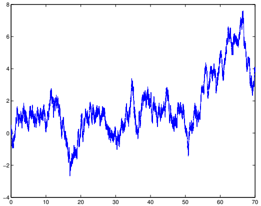

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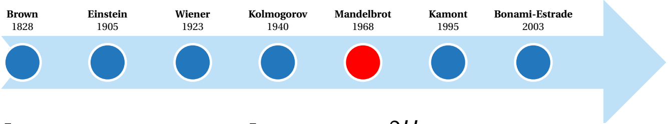
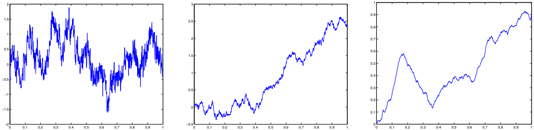


- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ ~~indpt. increments~~

fractional Brownian motion B^H (FBM)



$H = 0.2$ $\quad$ $H = 0.5$ $\quad$ $H = 0.8$

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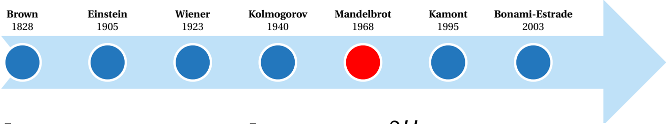

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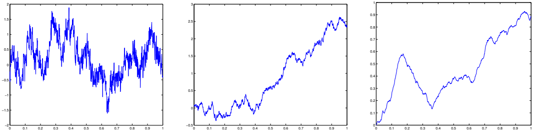

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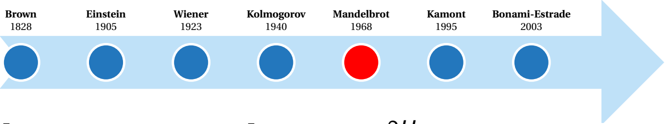

- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ stat. increments
- $\mathbf{R}(t,s) = \mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$

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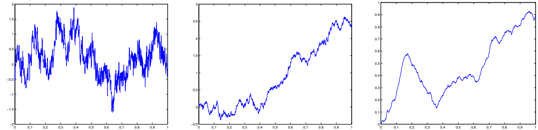

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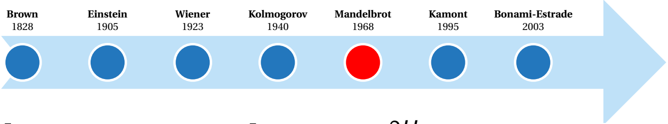

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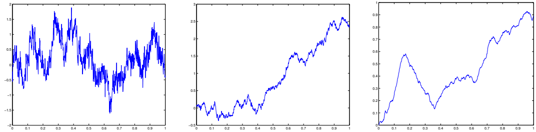
- $\mathbf{R}(t,s) = \mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$
- $B^H(t) = \frac{1}{c_H}\int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{j}t\xi}-1}{|\xi|^{H+1/2}}\widehat{\mathbf{W}}(\xi) \Rightarrow$ harmonizable formula

fractional Brownian motion B^H (FBM)



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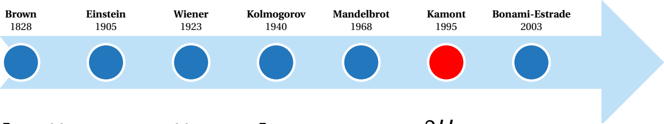
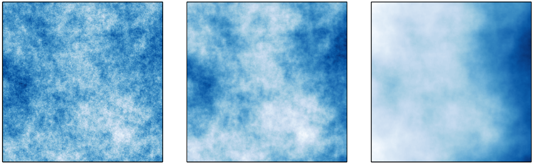


- $\mathbb{E}\left[(B^H(\boldsymbol{x}) - B^H(\boldsymbol{y}))^2\right] = \|\boldsymbol{x}-\boldsymbol{y}\|^{2H}$, $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^2$
- $\mathbf{R}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\left(\|\boldsymbol{x}\|^{2H} + \|\boldsymbol{y}\|^{2H} - \|\boldsymbol{x}-\boldsymbol{y}\|^{2H}\right)$
- $B^H(\boldsymbol{x}) = \frac{1}{C_H}\int_{\mathbb{R}^2} \frac{\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle}-1}{\|\boldsymbol{\xi}\|^{H+1}}\widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$

fractional Brownian field B^H (FBF)



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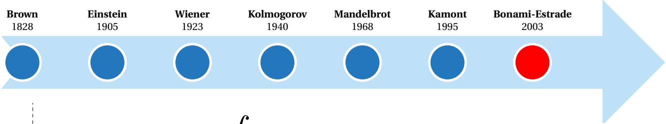
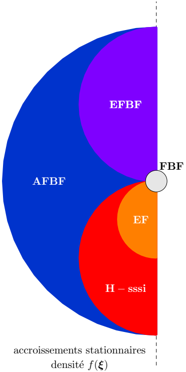
Model of Bonami-Estrade





$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{\xi}) \, \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

- $f^{1/2}(\boldsymbol{\xi}) = \dfrac{C}{\|\boldsymbol{\xi}\|^{H+1}}$ (FBF)
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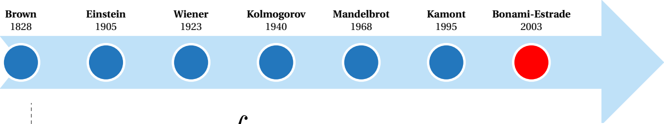

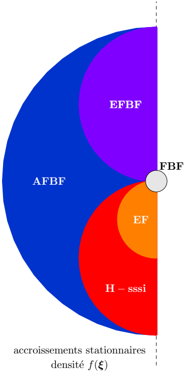

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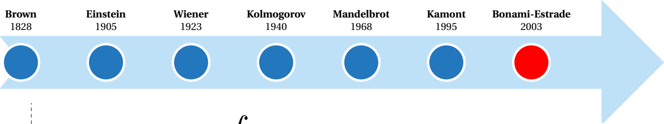

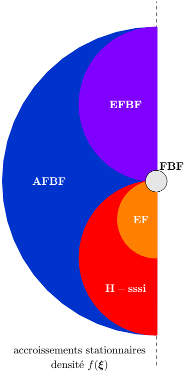

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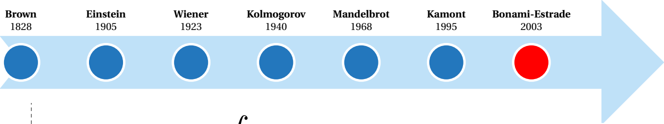

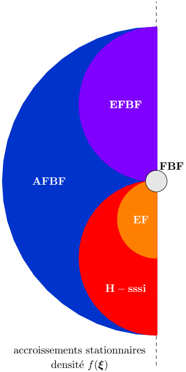

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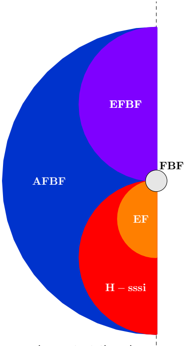
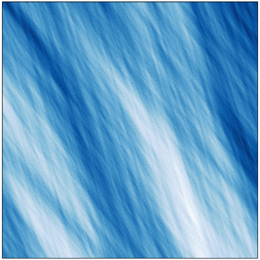
A special case of H-sssi: the elementary field

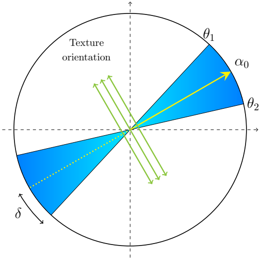
$$X(x) = \int_{\mathbb{R}^2} \left(e^{j\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1\right) \frac{\mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha_0)}{\|\boldsymbol{\xi}\|^{H+1}} \widehat{\mathbf{W}}(d\boldsymbol{\xi})$$



Elementary field (EF) $[H = 0.5,\ \alpha_0 = \pi/6]$



$\delta = 3.10^{-1}$



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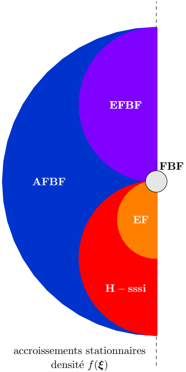

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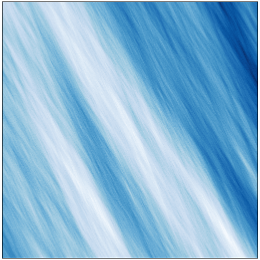

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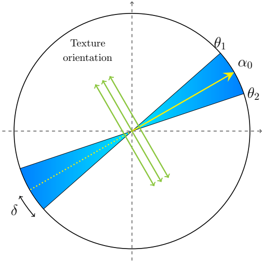

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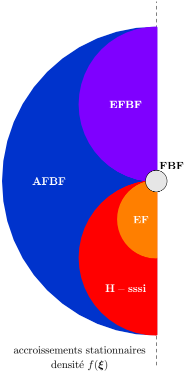

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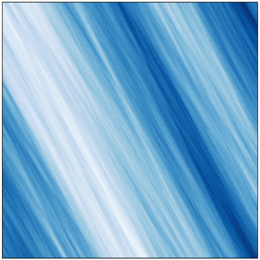

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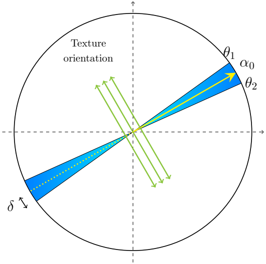

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State of the art: anisotropic Gaussian fields

- Fractional Brownian sheet (FBS) (Kamont, 1995), (Léger and Pontier, 1999), (Ayache et al., 2002)
- H-sssi fields (Benassi et coll., 1997)
- Model of Bonami and Estrade (Bonami and Estrade, 2003)
- Operator scaling Gaussian random fields (OSGRF) (Schertzer and Lovejoy, 1985), (Biermé et. al, 2007)
- Model of Xue, Xiao, Li (Xue and Xiao, 2011), (Li and Xiao, 2011)
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$\Rightarrow$ no class of fields with controlled local anisotropy

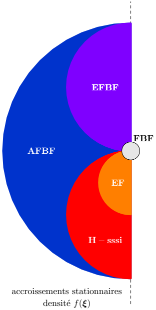
$\Rightarrow$ contribution : two new classes of this type
the (GAFBF) and the (WAFBF)

Two models: localized and warped H-sssi fields

From H-sssi fields to GAFBF

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{\xi}) \, \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

If X is H–self-similar, that is $X(\lambda \boldsymbol{x}) = \lambda^H X(\boldsymbol{x})$, one has:



H-sssi

$$f^{1/2}(\boldsymbol{\xi}) = \frac{C(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{H+1}}$$

EF

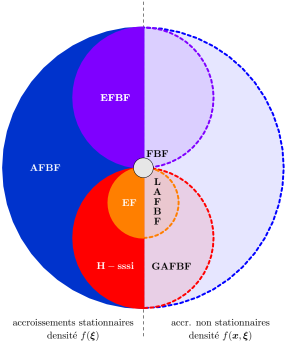
$$C(\boldsymbol{\xi}) = \mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha_0)$$

with homogeneous anisotropic function $\boldsymbol{\xi} \mapsto C(\boldsymbol{\xi})$

Model with prescribed orientations and regularities

New model: a localized and multifractional version of H-sssi fields

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1\right) f^{1/2}(\boldsymbol{x}, \boldsymbol{\xi}) \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$



GAFBF

$$f^{1/2}(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{C(\boldsymbol{x}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x})+1}}$$

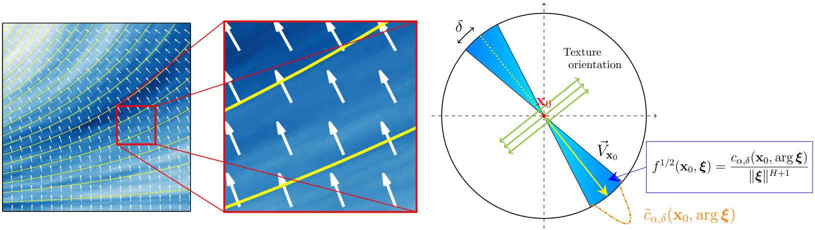
LAFBF

$$C(\boldsymbol{x}, \boldsymbol{\xi}) = \mathbb{1}_{[-\delta(\boldsymbol{x}), \delta(\boldsymbol{x})]}(\arg \boldsymbol{\xi} - \alpha(\boldsymbol{x}))$$

Model with prescribed local orientation

$$B^H_{\alpha,\delta}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(e^{j\langle \boldsymbol{x},\boldsymbol{\xi}\rangle} - 1\right) \frac{\mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\xi} - \alpha(\boldsymbol{x}))}{\|\boldsymbol{\xi}\|^{H+1}} \widehat{\mathbf{W}}(d\boldsymbol{\xi})$$

localized elementary field (LAFBF) [$H = 0.8$, $\alpha(x_1, x_2) = -\pi/2 + x_1$]



(a) (b) (c)

The tangent field: a tool for analysis and synthesis

1. A tool for analysis (Lévy-Vehel, 1995), (Falconer, 2002) :

$$\left\{\lim_{\rho\to 0}\frac{X(\boldsymbol{x_0}+\rho\boldsymbol{x})-X(\boldsymbol{x_0})}{\rho^{h(\boldsymbol{x}_0)}}\right\}_{\boldsymbol{x}\in\mathbb{R}^2}\overset{d}{=}\{Y_{\boldsymbol{x}_0}(\boldsymbol{x})\}_{\boldsymbol{x}\in\mathbb{R}^2}$$

Roughly speaking $Y_{\boldsymbol{x}_0}$ is the “local form” of X at point $\boldsymbol{x}_0$.

2. A tool for synthesis (Lévy-Véhel, 1995), (Benassi, 1997) :

$$X(\boldsymbol{x}_0)\leftarrow Y_{\boldsymbol{x}_0}(\boldsymbol{x}=\boldsymbol{x}_0)$$

⇒ If Y is “localizable”, all local anisotropy characteristics are defined and herited from its tangent field.

Assumptions on the GAFBF

Assumptions $(\mathcal{H})$

- h is β–Hölder, such that $a = \inf_{\boldsymbol{x}\in\mathbb{R}^2} h(\boldsymbol{x}) > 0$, $b = \sup_{\boldsymbol{x}\in\mathbb{R}^2} h(\boldsymbol{x})$ and $b < \beta \leq 1$.
- $(\boldsymbol{x}, \boldsymbol{\xi}) \mapsto C(\boldsymbol{x}, \boldsymbol{\xi})$ is bounded $C(\boldsymbol{x}, \boldsymbol{\xi}) \leqslant M, \forall(\boldsymbol{x}, \boldsymbol{\xi})$.
- $\boldsymbol{\xi} \mapsto C(\boldsymbol{x}, \boldsymbol{\xi})$ is even $C(\boldsymbol{x}, -\boldsymbol{\xi}) = C(\boldsymbol{x}, \boldsymbol{\xi})$.
- $\boldsymbol{\xi} \mapsto C(\boldsymbol{x}, \boldsymbol{\xi})$ homogeneous $C(\boldsymbol{x}, \rho\boldsymbol{\xi}) = C(\boldsymbol{x}, \boldsymbol{\xi}), \forall\rho$.
- $\boldsymbol{x} \mapsto C(\boldsymbol{x}, \boldsymbol{\xi})$ is continuous and $\exists\eta$, $\beta \leq \eta \leq 1$, $\forall \boldsymbol{x}$

$$\sup_{\boldsymbol{z}\in B(\mathbf{0},1)} \|\boldsymbol{z}\|^{-2\eta} \int_{\mathbb{S}^1} \left[C(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{\Theta}) - C(\boldsymbol{x}, \boldsymbol{\Theta})\right]^2 \mathrm{d}\boldsymbol{\Theta} \leq A_{\boldsymbol{x}} < \infty$$

Tangent field of the GAFBF

Let X be the GAFBF defined by

$$X(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C(\boldsymbol{x}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x})+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

Theorem (P. et al., 2017)

If X satisfies the assumptions $(\mathcal{H})$, then X admits at every point $\boldsymbol{x_0} \in \mathbb{R}^2$ a tangent field $Y_{\boldsymbol{x_0}}$ given by:

$$\begin{aligned} Y_{\boldsymbol{x_0}}(\boldsymbol{x}) &= \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{x_0}, \boldsymbol{\xi}) \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi}) , \\ &= \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C(\boldsymbol{x_0}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x_0})+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi}) . \end{aligned}$$

Tangent field of the GAFBF

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$$Y_{\boldsymbol{x_0}}(\boldsymbol{x}) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) f^{1/2}(\boldsymbol{x_0}, \boldsymbol{\xi}) \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi}) \,,$$

$$\text{H-sssi field} = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C_{\boldsymbol{x_0}}(\xi)}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x_0})+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi}) \,.$$

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2. A tool for synthesis (Lévy-Véhel, 1995), (Benassi, 1997) :

$$X(\mathbf{x}_0) \leftarrow Y_{\mathbf{x}_0}(\mathbf{x} = \mathbf{x}_0)$$

Multifractional Brownian field B^h (MBF) (Peltier, Vehel, 1995)

- Analysis : the MBF behaves locally as a FBF
$$\left\{ \lim_{\rho \to 0} \frac{B^h(\mathbf{x_0} + \rho \mathbf{x}) - B^h(\mathbf{x_0})}{\rho^{h(\mathbf{x}_0)}} \right\}_{\mathbf{x} \in \mathbb{R}^2} \stackrel{d}{=} \left\{ B^{h(\mathbf{x}_0)}(\mathbf{x}) \right\}_{\mathbf{x} \in \mathbb{R}^2}$$
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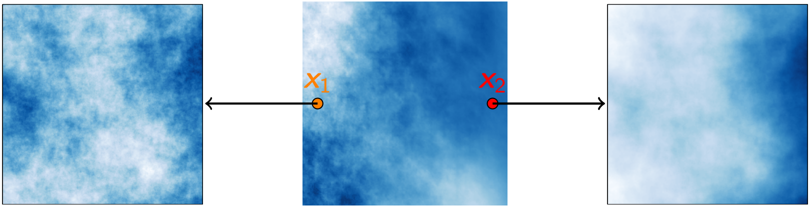
B^H, $H \equiv h(\boldsymbol{x}_1)$ MBM $B^{h(\boldsymbol{x})}$ B^H, $H \equiv h(\boldsymbol{x}_2)$



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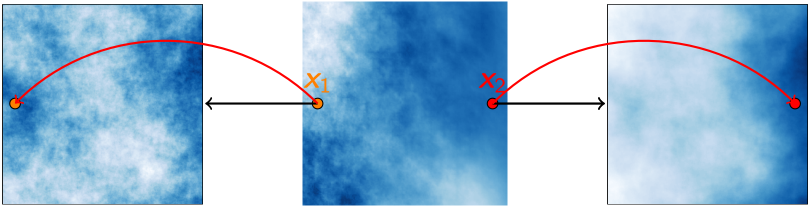
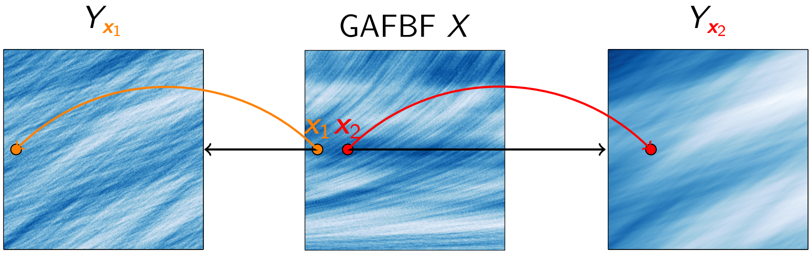
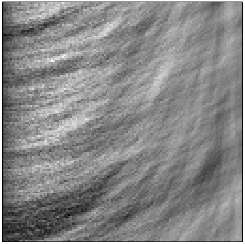


Synthesis of the GAFBF by its tangent fields

$$X(\boldsymbol{x}_0) \leftarrow Y_{\boldsymbol{x}_0}(\boldsymbol{x} = \boldsymbol{x}_0) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} - 1) \frac{C_{\boldsymbol{x}_0}(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{h(\boldsymbol{x}_0)+1}} \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$



Simulation of the LAFBF



- Linear variation of the orientations $\alpha(\boldsymbol{x})$ along (Ox)
- Linear variation of the directionality $\delta(\boldsymbol{x})$ along (Ox)
- Linear variation of the regularity $h(\boldsymbol{x})$ along (Ox)

The WAFBF: warped H-sssi fields

Definition (WAFBF)

Let X be a H-sssi field and $\boldsymbol{\Phi} : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function. The *Warped Anisotropic Fractional Brownian Field* (WAFBF) $Z_{\boldsymbol{\Phi},X}$ is defined as the deformation of the elementary field X by the application $\boldsymbol{\Phi}$:

$$Z_{\boldsymbol{\Phi},X}(\boldsymbol{x}) = X(\boldsymbol{\Phi}(\boldsymbol{x})) \ .$$

References about deformations of stationary random fields:

- (Perrin and Senoussi, 1999, 2000)
- (Guyon and Perrin, 2000)

The WAFBF: warped H-sssi fields

Definition (WAFBF)

Let X be a H-sssi field and $\mathbf{\Phi} : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function. The WAFBF $Z_{\mathbf{\Phi},X}$ is defined as the deformation of the elementary field X by the application $\mathbf{\Phi}$:

$$Z_{\mathbf{\Phi},X}(\boldsymbol{x}) = X(\mathbf{\Phi}(\boldsymbol{x})) \ .$$

Theorem (Tangent field of the WAFBF)

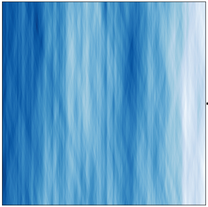
$Z_{\mathbf{\Phi},X}$ admits at every point $\boldsymbol{x_0} \in \mathbb{R}^2$ the tangent field:

$$Y_{\boldsymbol{x_0}}(\boldsymbol{x}) = X(\mathbf{D\Phi}(\boldsymbol{x_0})\ \boldsymbol{x}) \ , \quad \forall \boldsymbol{x} \in \mathbb{R}^2 \ ,$$

where $\mathbf{D\Phi}(\boldsymbol{x_0})$ is the jacobian matrix of $\mathbf{\Phi}$ at point $\boldsymbol{x_0}$.

Warped elementary field

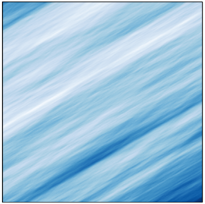
$$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$$



X

$$\mathbf{\Phi}(x) \longrightarrow \mathbf{R}_{-\alpha(\mathbf{x})} x$$

WAFBF

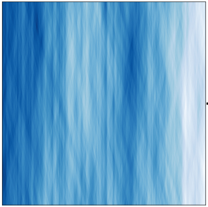


$Z = X \circ \mathbf{\Phi}$

$$\alpha(x_1, x_2) = -\frac{\pi}{4}$$

Warped elementary field

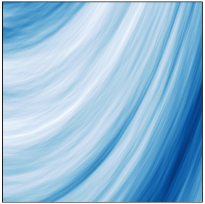
$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$



X

$\xrightarrow[\mathbf{R}_{-\alpha(\mathbf{x})}\mathbf{x}]{\mathbf{\Phi}(\mathbf{x})}$

WAFBF

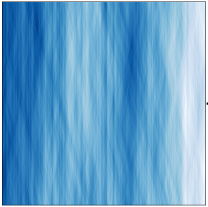


$Z = X \circ \mathbf{\Phi}$

$$\alpha(x_1, x_2) = -\frac{\pi}{2} + x_1$$

Warped elementary field

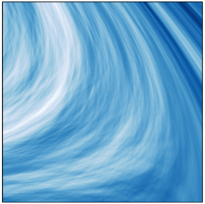
$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$



X

$\mathbf{\Phi}(x)$ → $\mathbf{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}$

WAFBF

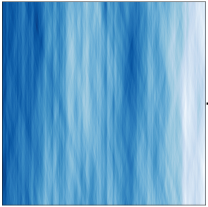


$Z = X \circ \mathbf{\Phi}$

$$\alpha(x_1, x_2) = -\frac{\pi}{2} + x_2$$

Warped elementary field

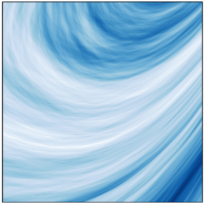
$$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$$



X

$$\xrightarrow[\mathbf{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}]{\boldsymbol{\Phi}(\boldsymbol{x})}$$

WAFBF

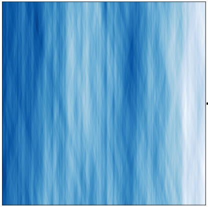


$Z = X \circ \boldsymbol{\Phi}$

$$\alpha(x_1, x_2) = -\frac{\pi}{2} + x_1^2 - x_2$$

Warped elementary field

$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$

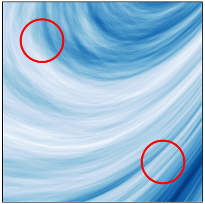


X

$\boldsymbol{\Phi}(x)$

$\mathbf{R}_{-\alpha(x)} x$

WAFBF

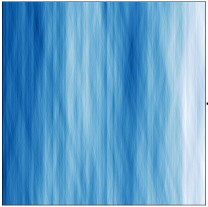


$Z = X \circ \boldsymbol{\Phi}$

1. The directionnality is not controlled

Warped elementary field

$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$

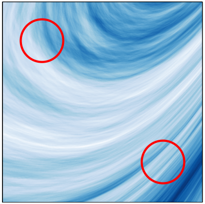


X

$\mathbf{\Phi}(\boldsymbol{x})$

$\mathbf{R}_{-\alpha(\boldsymbol{x})}\boldsymbol{x}$

WAFBF

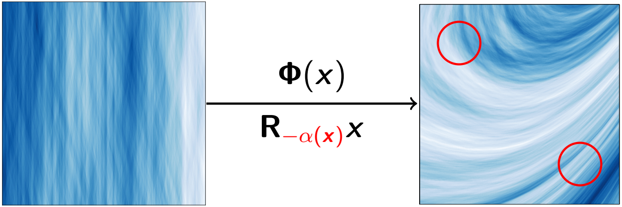


$Z = X \circ \mathbf{\Phi}$

1. The directionnality is not controlled
2. What transformation $\mathbf{\Phi}$ makes it possible to prescribe the orientation at each point $\alpha(\boldsymbol{x})$?

Warped elementary field

$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$ WAFBF



X $Z = X \circ \mathbf{\Phi}$

1. The directionnality is not controlled
2. What transformation $\mathbf{\Phi}$ makes it possible to prescribe the orientation at each point $\alpha(\boldsymbol{x})$?
3. What definition for the orientation of a random field ?

Definition of the notion of orientation for random fields

Local orientation of a deterministic function

Gradient operator

The gradient operator $\nabla : f \mapsto (\partial_{x_1} f, \partial_{x_2} f)$, with the notation $\partial_{x_p} f : \boldsymbol{x} = (x_1, x_2) \mapsto \frac{\partial f}{\partial x_p}(\boldsymbol{x})$, is defined in Fourier domain by:

$$\widehat{\partial_{x_1} f}(\boldsymbol{\omega}) = -\mathrm{j}\omega_1 \widehat{f}(\boldsymbol{\omega}), \quad \widehat{\partial_{x_2} f}(\boldsymbol{\omega}) = -\mathrm{j}\omega_2 \widehat{f}(\boldsymbol{\omega})$$

$\Rightarrow$ Orientation: $\boldsymbol{n}(\boldsymbol{x}) = \dfrac{\nabla f(\boldsymbol{x})}{\|\nabla f(\boldsymbol{x})\|}$, $\theta(\boldsymbol{x}) = \arctan\left(\dfrac{\partial_{x_2} f(\boldsymbol{x})}{\partial_{x_1} f(\boldsymbol{x})}\right)$

Local orientation of a deterministic function

Gradient operator

The gradient operator $\nabla : f \mapsto (\partial_{x_1} f, \partial_{x_2} f)$, with the notation $\partial_{x_p} f : \boldsymbol{x} = (x_1, x_2) \mapsto \frac{\partial f}{\partial x_p}(\boldsymbol{x})$, is defined in Fourier domain by:

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$\Rightarrow$ (More robust) minimize the directions against ∇f:

$$\max_{\theta'} \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \left\langle \boldsymbol{n}(\theta'), \nabla f(\boldsymbol{x}') \right\rangle^2 \mathrm{d}\boldsymbol{x}' = \max_{\theta'} \boldsymbol{n}(\theta')^{\mathsf{T}} \mathbf{J}_f^W(\boldsymbol{x}) \boldsymbol{n}(\theta')$$

$$[\mathbf{J}_f^W(\boldsymbol{x})]_{pq} = \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \partial_{x_p} f(\boldsymbol{x}') \partial_{x_q} f(\boldsymbol{x}') \,\mathrm{d}\boldsymbol{x}', \quad p, q \in \{1, 2\}$$

Local orientation of a deterministic function

Riesz transform and monogenic signal (Felsberg, 2001)

The Riesz operator $\boldsymbol{\mathcal{R}} : f \mapsto (\mathcal{R}_1 f, \mathcal{R}_2 f)$ is defined by:

$$\widehat{\mathcal{R}_1 f}(\boldsymbol{\omega}) = -\mathrm{j}\frac{\omega_1}{\|\boldsymbol{\omega}\|}\widehat{f}(\boldsymbol{\omega}), \quad \widehat{\mathcal{R}_2 f}(\boldsymbol{\omega}) = -\mathrm{j}\frac{\omega_2}{\|\boldsymbol{\omega}\|}\widehat{f}(\boldsymbol{\omega})$$

$\Rightarrow$ Orientation: $\boldsymbol{n}(\boldsymbol{x}) = \dfrac{\boldsymbol{\mathcal{R}} f(\boldsymbol{x})}{\|\boldsymbol{\mathcal{R}} f(\boldsymbol{x})\|}$, $\theta(\boldsymbol{x}) = \arctan\left(\dfrac{\mathcal{R}_2 f(\boldsymbol{x})}{\mathcal{R}_1 f(\boldsymbol{x})}\right)$

Local orientation of a deterministic function

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f $\quad$ $\mathcal{R}_1 f$ $\quad$ $\mathcal{R}_2 f$ $\quad$ $\|\boldsymbol{\mathcal{R}} f\|$ $\quad$ θ

Credits: R. Soulard

Local orientation of a deterministic function

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$\Rightarrow$ (More robust) minimize the directions against $\boldsymbol{\mathcal{R}} f$:

$$\max_{\theta'} \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \left\langle \boldsymbol{n}(\theta'), \boldsymbol{\mathcal{R}} f(\boldsymbol{x}') \right\rangle^2 \mathrm{d}\boldsymbol{x}' = \max_{\theta'} \boldsymbol{n}(\theta')^{\mathsf{T}} \mathbf{J}_f^W(\boldsymbol{x}) \boldsymbol{n}(\theta')$$

$$[\mathbf{J}_f^W(\boldsymbol{x})]_{pq} = \int_{\mathbb{R}^2} w(\boldsymbol{x}-\boldsymbol{x}') \mathcal{R}_p f(\boldsymbol{x}') \mathcal{R}_q f(\boldsymbol{x}') \, \mathrm{d}\boldsymbol{x}', \quad p, q \in \{1, 2\}$$

Local orientation of a deterministic function

Structure tensor

The structure tensor $\mathbf{J}_f^w(\boldsymbol{x}) = \mathbf{J}_f(\boldsymbol{x}) * w$ is defined from following symmetric matrix, positive definite and of rank one:

$$\mathbf{J}_f(\boldsymbol{x}) = \boldsymbol{\mathcal{R}} f(\boldsymbol{x}) \boldsymbol{\mathcal{R}} f(\boldsymbol{x})^{\mathsf{T}} = \begin{pmatrix} \mathcal{R}_1 f(\boldsymbol{x})^2 & \mathcal{R}_1 f(\boldsymbol{x}) \mathcal{R}_2 f(\boldsymbol{x}) \\ \mathcal{R}_1 f(\boldsymbol{x}) \mathcal{R}_2 f(\boldsymbol{x}) & \mathcal{R}_2 f(\boldsymbol{x})^2 \end{pmatrix}$$

Local orientation & coherency index

- The local orientation $\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{\mathcal{R}} f(\boldsymbol{x}) / \|\boldsymbol{\mathcal{R}} f(\boldsymbol{x})\|$ of f at point $\boldsymbol{x}$ corresponds to the unit eigenvector associated to the largest of the eigenvalues $\lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x})$ of $\mathbf{J}_f^w(\boldsymbol{x})$
- The coherence index provides a degree of directionality:

$$\chi_f(\boldsymbol{x}) = \frac{|\lambda_2(\boldsymbol{x}) - \lambda_1(\boldsymbol{x})|}{\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})}$$

Global definition of orientation for H-sssi fields

Structure tensor in the self-similar stationary case

Let X be a H-sssi field whose anisotropy function C_X is bounded and ψ a zero-mean isotropic window admitting two vanishing moments.

We define the following random structure tensor:

$$\mathbf{J}_X^{\psi} = \begin{pmatrix} |\langle X, \mathcal{R}_1\psi\rangle|^2 & \langle X, \mathcal{R}_1\psi\rangle\overline{\langle X, \mathcal{R}_2\psi\rangle} \\ \langle X, \mathcal{R}_1\psi\rangle\overline{\langle X, \mathcal{R}_2\psi\rangle} & |\langle X, \mathcal{R}_1\psi\rangle|^2 \end{pmatrix}$$

with the Gaussian variable $\langle X, \mathcal{R}_\ell\psi\rangle = \int_{\mathbb{R}^2} X(\boldsymbol{x})\mathcal{R}_\ell\psi(\boldsymbol{x})\,\mathrm{d}x$

Global definition of orientation for H-sssi fields

Theorem (P. et al., 2019)

Define $\widehat{\psi}(\boldsymbol{\xi}) = \varphi(\|\boldsymbol{\xi}\|)$. Then

$$\mathbb{E}\left[\mathbf{J}_X^{\psi}\right] = \left(\int_0^{+\infty} \frac{|\varphi(r)|^2}{r^{2H+1}}\,\mathrm{d}r\right) \mathbf{J}_X$$

where $\mathbf{J}_X$ is called the tensor structure of X defined by :

$$[\mathbf{J}_X]_{\ell_1\ell_2} = \int_{\boldsymbol{\Theta}\in\mathbb{S}^1} \Theta_{\ell_1}\Theta_{\ell_2}\ C_X(\boldsymbol{\Theta})^2\,\mathrm{d}\boldsymbol{\Theta}, \quad \ell_1,\ell_2 \in \{1,2\}\ .$$

Global definition of orientation for H-sssi fields

Definition (Orientation & coherence index of a H-sssi field X)

- The orientation $\vec{\boldsymbol{n}}_X$ of X is given by the unit eigenvector associated to the largest of the eigenvalues λ_1, λ_2 of $\mathbf{J}_X$
- The coherence index of X is defined by

$$\chi = \frac{|\lambda_2 - \lambda_1|}{\lambda_1 + \lambda_2}$$

Orientations of an elementary field (EF)

Orientation of an elementary field

$X = X_{\alpha_0,\delta}$ with $C_X(\boldsymbol{\Theta}) = \mathbb{1}_{[-\delta,\delta]}(\arg \boldsymbol{\Theta} - \alpha_0)$

$$\vec{\boldsymbol{n}}_X = \boldsymbol{u}(\alpha_0) = \begin{pmatrix} \cos \alpha_0 \\ \sin \alpha_0 \end{pmatrix}, \quad \chi(X) = \frac{\sin(2\delta)}{2\delta}$$

$$\mathbf{J}_X = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\cos(\alpha_0)\frac{\sin(2\delta)}{2\delta} & \frac{1}{2}\sin(\alpha_0)\frac{\sin(2\delta)}{2\delta} \\ \frac{1}{2}\sin(\alpha_0)\frac{\sin(2\delta)}{2\delta} & \frac{1}{2} - \frac{1}{2}\cos(\alpha_0)\frac{\sin(2\delta)}{2\delta} \end{pmatrix}$$

Orientations of an elementary field (EF)

Orientation of an elementary field

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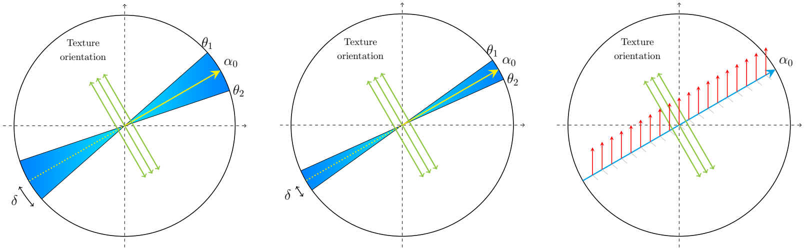


Linear transformation of an EF and its orientation

Sum of two independant elementary fields

Let define $X = X_{\alpha_0,\delta} + X_{\alpha_1,\delta}$ the sum of two independant EF.

$$\vec{\boldsymbol{n}}_X = \boldsymbol{u}\left(\frac{\alpha_0 + \alpha_1}{2}\right), \quad \chi(X) = \frac{\sin(2\delta)}{2\delta}\cos(\alpha_0 - \alpha_1)$$

Deformation of an elementary field

Let $\mathbf{L}$ be an invertible 2×2 matrix and $X_{\mathbf{L}}(\boldsymbol{x}) = X_{\alpha_0,\delta}(\mathbf{L}\boldsymbol{x})$

$$\vec{\boldsymbol{n}}_{X_{\mathbf{L}}} = \frac{\mathbf{L}^{\mathsf{T}}\boldsymbol{u}(\alpha_0)}{\|\mathbf{L}^{\mathsf{T}}\boldsymbol{u}(\alpha_0)\|}$$

Orientation of a localizable Gaussian field

Localizable Gaussian field

A random field $X = \{X(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^2\}$ is said to be localizable, if it admits a tangent field at every point $\boldsymbol{x} \in \mathbb{R}^2$.

References : (Lévy-Véhel, 1995), (Benassi et coll., 1997), (Falconer, 2002).

Definition (Local orientation of a localizable Gaussian field)

The local orientation $\vec{\boldsymbol{n}}_X(\boldsymbol{x}_0)$ of the localizable Gaussian field X at point $\boldsymbol{x_0}$ is the orientation of its tangent field $Y_{\boldsymbol{x_0}}$ H-sssi :

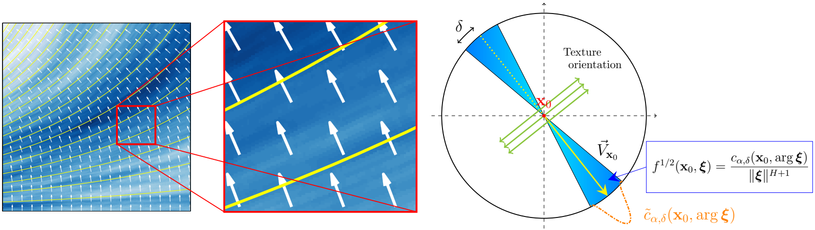
$$\vec{\boldsymbol{n}}_X(\boldsymbol{x}_0) \equiv \vec{\boldsymbol{n}}_{Y_{\boldsymbol{x}_0}}$$

Orientation of a localizable Gaussian field

Local orientation of the LAFBF X

The local orientation $\vec{n}_X(x_0)$ and the coherence index $\chi(x_0)$ of X at x_0 are those of the elementary field $X_{\alpha(x_0),\delta(x_0)}$:

$$\vec{n}_X(x_0) \equiv \vec{n}_{X_{\alpha_0(x_0),\delta(x_0)}} = \begin{pmatrix} \cos\alpha(x_0) \\ \sin\alpha(x_0) \end{pmatrix}, \quad \chi(x_0) = \frac{\sin(2\delta(x_0))}{2\delta(x_0)}$$



(a) (b) (c)

Orientation of a localizable Gaussian field

Local orientation of the WAFBF where $X = X_{\alpha_0,\delta}$ is an EF

The tangent field of $Z_{\mathbf{\Phi},X}(\boldsymbol{x}) = X_{\alpha_0,\delta}(\mathbf{\Phi}(\boldsymbol{x}))$ at $\boldsymbol{x}_0$ is

$$Y_{\boldsymbol{x}_0}(\boldsymbol{x}) = X_{\alpha_0,\delta}(\mathbf{D\Phi}(\boldsymbol{x}_0)\ \boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^2$$

whose orientation is $\vec{\boldsymbol{n}}_{Y_{\boldsymbol{x}_0}} = \frac{\mathbf{L}^\mathsf{T}\boldsymbol{u}(\alpha_0)}{\|\mathbf{L}^\mathsf{T}\boldsymbol{u}(\alpha_0)\|}$ with $\mathbf{L} = \mathbf{D\Phi}(\boldsymbol{x}_0)$, hence

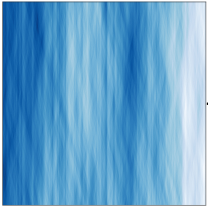
$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x}_0) \equiv \vec{\boldsymbol{n}}_{Y_{\boldsymbol{x}_0}} = \frac{\mathbf{D\Phi}(\boldsymbol{x}_0)^\mathsf{T}\boldsymbol{u}(\alpha_0)}{\|\mathbf{D\Phi}(\boldsymbol{x}_0)^\mathsf{T}\boldsymbol{u}(\alpha_0)\|}$$

Orientation of a localizable Gaussian field

Rotation locale du WAFBF où $X = X_{0,\delta}$

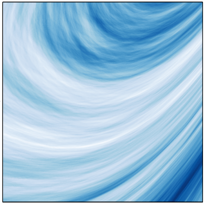
The local orientation of $Z_{\boldsymbol{\Phi}_\alpha, X}(x) = X_{0,\delta}(\boldsymbol{\Phi}_\alpha(x))$ at x_0 with $\boldsymbol{\Phi}_\alpha(x) = \mathbf{R}_{-\alpha(x)} x$ is given by $\vec{n}_Z(x_0) = \frac{\mathbf{D}\boldsymbol{\Phi}(x_0)^T \boldsymbol{e}_1}{\|\mathbf{D}\boldsymbol{\Phi}(x_0)^T \boldsymbol{e}_1\|}$, that is :

$$\vec{n}_Z(x_0) = \frac{\boldsymbol{u}(\alpha(x_0)) + \langle \boldsymbol{u}(\alpha(x_0))^\perp, x_0 \rangle \nabla\alpha(x_0)}{\|\boldsymbol{u}(\alpha(x_0)) + \langle \boldsymbol{u}(\alpha(x_0))^\perp, x_0 \rangle \nabla\alpha(x_0)\|}$$



$X_{0,\delta}$

$\boldsymbol{\Phi}(x)$ → $\mathbf{R}_{-\alpha(x)} x$



$Z_{\boldsymbol{\Phi}, X} = X_{0,\delta} \circ \boldsymbol{\Phi}$

Prescribed orientations for the WAFBF

Proposition (Orientation controled by harmonic functions)

Let $Z_{\boldsymbol{\Phi}_\alpha, X}(\boldsymbol{x})$ be the field $X = X_{0,\delta}$ of orientation $\boldsymbol{e}_1 = (1,0)^T$ warped by a conform transformation $\boldsymbol{\Phi}_\alpha$ defined by:

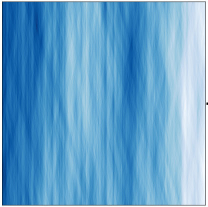
1. $\alpha : \mathbb{R}^2 \to \mathbb{R}$ a harmonic function
2. λ its conjugate harmonic function such as $\boldsymbol{\Psi}_\alpha = \begin{pmatrix} \lambda \\ -\alpha \end{pmatrix}$ is holomorphic
3. $\boldsymbol{\Phi}_\alpha$ a complex primitive of $\exp(\boldsymbol{\Psi}_\alpha)$

The local orientation (up to δ^2) of $Z_{\boldsymbol{\Phi}_\alpha, X}$ at $\boldsymbol{x_0}$ is

$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x_0}) = \begin{pmatrix} \cos\alpha(\boldsymbol{x_0}) \\ \sin\alpha(\boldsymbol{x_0}) \end{pmatrix} = \boldsymbol{u}(\alpha(\boldsymbol{x}_0))$$

WAFBF with prescribed local orientations

$C(\xi) = \mathbb{1}_{[-\delta,\delta]}(\arg \xi)$

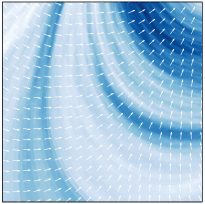


$X_{0,\delta}$

$\mathbf{\Phi}_\alpha$

$\alpha(x_1, x_2) = ax_1 + bx_2 + c$

WAFBF $(a, b) = (2, -1)$



$Z = X_{0,\delta} \circ \mathbf{\Phi}_\alpha$

$$\mathbf{\Phi}_\alpha(x_1, x_2) = \frac{\exp(ax_2 - bx_1)}{a^2 + b^2} \begin{pmatrix} a\sin(ax_1 + bx_2 + c) - b\cos(ax_1 + bx_2 + c) \\ a\cos(ax_1 + bx_2 + c) + b\sin(ax_1 + bx_2 + c) \end{pmatrix}$$

$$\vec{\boldsymbol{n}}_Z(\boldsymbol{x}) = \frac{\mathbf{D\Phi}(\boldsymbol{x})^{\mathsf{T}}(1,0)}{\|\mathbf{D\Phi}(\boldsymbol{x})^{\mathsf{T}}(1,0)\|} = (\cos\alpha(\boldsymbol{x}), \sin\alpha(\boldsymbol{x}))$$

Conclusion

Conclusion

- Two models of anisotropic Gaussian fields enabling to control the local orientation at every point
- Definition of a global orientation for the self-similar case
- Turn the global definition to a local one by considering localizable fields behaving locally as self-similar ones
- Show the consistency of our approach on our two models

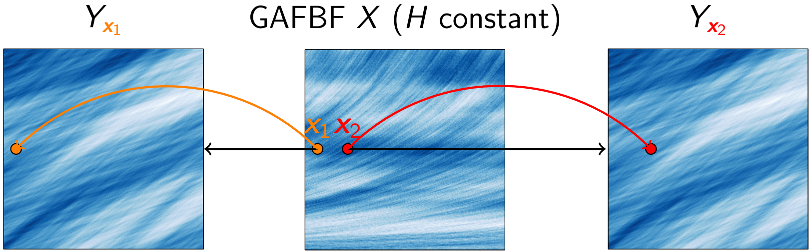
Perspectives

- Definition of the Riesz transform for random fields
- Estimation of the roughness and anisotropy by wavelets
- Test hypothesis for the directionality of a texture

Synthesis of GAFBF inspired from (Wood, 1994)

1. Simulate U GAFBF X^{H_u} with constant regularities $(H_u)_{1\leq u\leq U}$:

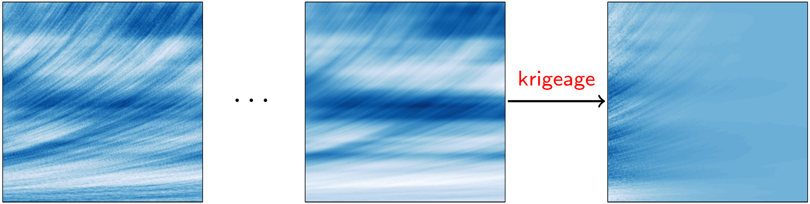
$$X^{H_u}(\boldsymbol{x}_0) \leftarrow Y_{\boldsymbol{x}_0}(\boldsymbol{x}=\boldsymbol{x}_0) = \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle} - 1)\frac{C_{\boldsymbol{x}_0}(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^{H_u+1}}\widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$



Synthesis of GAFBF inspired from (Wood, 1994)

2. Simulate the GAFBF with variable regularity by krigeage :
Spatial interpolation of the (X^{H_u}) from the covariance

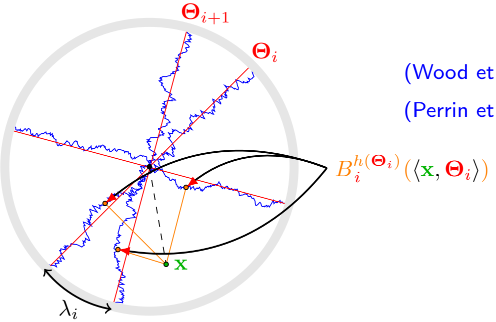
GAFBF X^{H_1} … GAFBF X^{H_U} → GAFBF X



Synthesis of H-sssi fields by turning bands

$$Y_{\boldsymbol{x}_0}^{[n]}(\boldsymbol{x}) = \sum_{i=1}^{n} \omega_i(\boldsymbol{x}_0) B_i^H(\langle \boldsymbol{x}, \boldsymbol{\Theta}_i \rangle) \ ,$$

$\Rightarrow$ Simulate n FBM B_i^H of complexity $O(\ell \log \ell)$



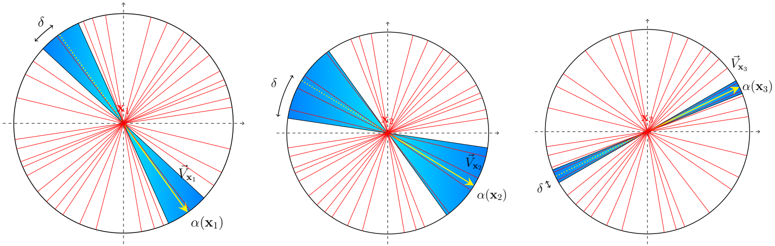
(Wood et coll., 1994)

(Perrin et coll., 2002)

Simulation of the LAFBF with H constant

$$B^H_{\alpha,\delta}(x_0) \leftarrow Y^{[n]}_{x_0}(x = x_0) = \sum_{i=1}^{n} \omega_i(x_0) B_i^H(\langle x_0, \Theta_i \rangle) \ ,$$

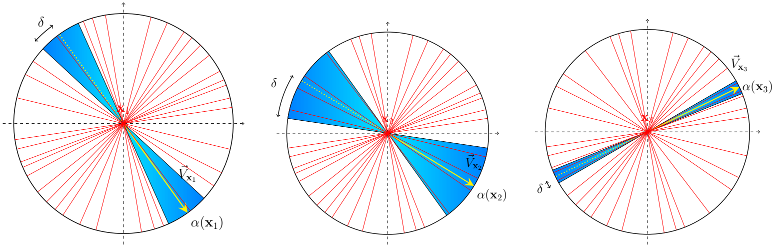
$$\omega_i(x_0)^2 \propto C_{x_0}(\Theta_i) = \mathbb{1}_{[-\delta(x_0),\delta(x_0)]}(\arg \Theta_i - \alpha(x_0))$$



Simulation of the LAFBF with H constant

$$B^H_{\alpha,\delta}(\boldsymbol{x}_0) \leftarrow Y^{[n]}_{\boldsymbol{x}_0}(\boldsymbol{x}=\boldsymbol{x}_0) = \sum_{i=1}^{n} \omega_i(\boldsymbol{x}_0) B_i^H(\langle \boldsymbol{x}_0, \boldsymbol{\Theta}_i \rangle) \ ,$$

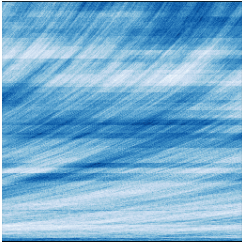
- Pre-computing of the n B_i^H (complexity $O(\ell \log \ell)$)
- The rest of the algorithm is of complexity $O(\log n\#\text{pixels})$



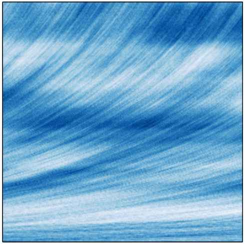
Simulation of the LAFBF with H constant

$$B_{\alpha,\delta}^{H} \leftarrow Y_{\mathbf{x}_0}^{[n]}(\mathbf{x}=\mathbf{x}_0) = \sum_{i=1}^{n} \omega_i(\mathbf{x}_0) B_i^H \left(\langle \mathbf{x}_0, \, \boldsymbol{\Theta}_i \rangle\right) \ ,$$

$$\omega_i(\mathbf{x}_0)^2 \propto C_{\mathbf{x}_0}(\boldsymbol{\Theta}_i) = \mathbb{1}_{[-\delta(\mathbf{x}_0),\delta(\mathbf{x}_0)]}(\arg \boldsymbol{\Theta}_i - \alpha(\mathbf{x}_0))$$



$C_{\mathbf{x}_0}(\boldsymbol{\Theta}_i)$

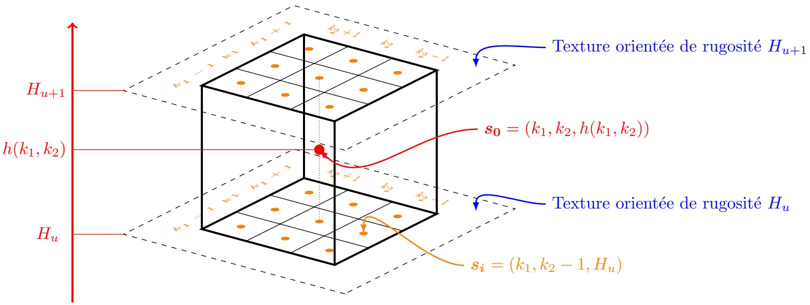


$\widetilde{C}_{\mathbf{x}_0}(\boldsymbol{\Theta}_i)$ régularisée

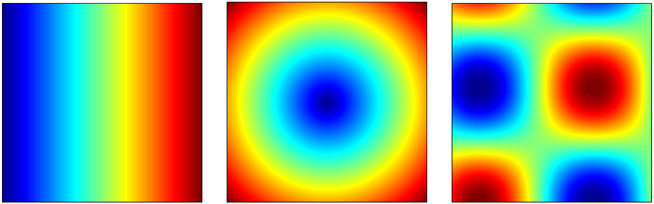
Simulation of the LAFBF with h variable (krigeage)

$$\widehat{Z}(\boldsymbol{s_0}) = \sum_{i \in \mathcal{V}(\boldsymbol{s_0})} \lambda_i Z(\boldsymbol{s_i}) = \boldsymbol{\lambda}^\mathsf{T} \boldsymbol{Z} \quad (\text{BLUE})$$

$$Z = B^{h}_{\alpha,\delta},\ (B^{H_u}_{\alpha,\delta})_{1 \le u \le U} \to Z(\boldsymbol{s_i}),\ \boldsymbol{\Sigma}_{ij} = \mathrm{Cov}(Z(\boldsymbol{s_i}), Z(\boldsymbol{s_j})) \to \boldsymbol{\lambda}$$



Simulation of the LAFBF



h linear | h radial | h sinusoidal

Local orientation of a deterministic function

Monogenic wavelet coefficients (Unser, Olhede, 2009)

Let $\psi_{i,\boldsymbol{k}}(\boldsymbol{x}) = 2^i \psi(2^i \boldsymbol{x} - \boldsymbol{k})$ be a wavelet frame constructed from a real isotropic wavelet $\widehat{\psi}(\boldsymbol{\xi}) = \varphi(\|\boldsymbol{\xi}\|)$. We consider the wavelet coefficients of $\boldsymbol{\mathcal{R}} f$ in the frame $\{\psi_{i,\boldsymbol{k}}\}$:

$$c_{i,\boldsymbol{k}}^{(\mathcal{R})}(f) = \begin{pmatrix} c_{i,\boldsymbol{k}}^{(1)}(f) \\ c_{i,\boldsymbol{k}}^{(2)}(f) \end{pmatrix} = \begin{pmatrix} \langle \mathcal{R}_1 f, \psi_{i,\boldsymbol{k}} \rangle \\ \langle \mathcal{R}_2 f, \psi_{i,\boldsymbol{k}} \rangle \end{pmatrix} = \begin{pmatrix} \langle f, \mathcal{R}_1 \psi_{i,\boldsymbol{k}} \rangle \\ \langle f, \mathcal{R}_2 \psi_{i,\boldsymbol{k}} \rangle \end{pmatrix}$$

Tensor structure of the wavelet coefficients:

$$\mathbf{J}_{f,i}^{W}[\boldsymbol{k}] = c_{i,\boldsymbol{k}}^{(\mathcal{R})}(f) c_{i,\boldsymbol{k}}^{(\mathcal{R})}(f)^* = \begin{pmatrix} \left|c_{i,\boldsymbol{k}}^{(1)}(f)\right|^2 & c_{i,\boldsymbol{k}}^{(1)}(f) \cdot \overline{c_{i,\boldsymbol{k}}^{(2)}(f)} \\ \overline{c_{i,\boldsymbol{k}}^{(1)}(f)} \cdot c_{i,\boldsymbol{k}}^{(2)}(f) & \left|c_{i,\boldsymbol{k}}^{(1)}(f)\right|^2 \end{pmatrix}$$

Orientation of a H-sssi field

Monogenic wavelet coefficients of a H-sssi field X

$$c_{i,\mathbf{k}}^{(\ell)}(X) = \langle X, \mathcal{R}_\ell \psi_{i,\mathbf{k}} \rangle = \int_{\mathbb{R}^2} \widehat{\mathcal{R}_\ell \psi_{i,\mathbf{k}}}(\boldsymbol{\xi}) C_X(\boldsymbol{\xi}) \, \|\boldsymbol{\xi}\|^{-H-1} \, \widehat{\mathbf{W}}(\mathrm{d}\boldsymbol{\xi})$$

Theorem (P. et al., 2017)

Let us define $c_{i,\mathbf{k}}^{(\mathcal{R})}(X) = (c_{i,\mathbf{k}}^{(1)}(X), c_{i,\mathbf{k}}^{(2)}(X))^{\mathsf{T}}$, then:

$$\mathbb{E}[c_{i,\mathbf{k}}^{(\mathcal{R})}(X) c_{i,\mathbf{k}}^{(\mathcal{R})}(X)^*] \propto 2^{-2i(H+1)} \mathbf{J}_X \ ,$$

where $\mathbf{J}_X$ is called the tensor structure of X defined by :

$$[\mathbf{J}_X]_{\ell_1 \ell_2} = \int_{\boldsymbol{\Theta} \in \mathbb{S}^1} \Theta_{\ell_1} \Theta_{\ell_2} \ C(\boldsymbol{\Theta})^2 \, \mathrm{d}\boldsymbol{\Theta}, \quad \ell_1, \ell_2 \in \{1, 2\} \ .$$