Une approche convexe de la super-résolution et de la régularisation de lignes 2D dans les images

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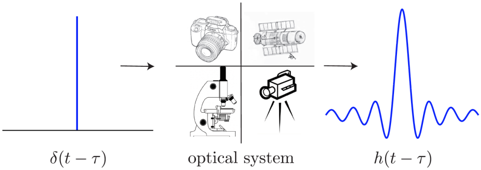
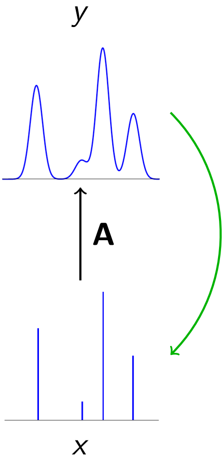
Super-resolution of 1-D impulses



$$x(t) = \sum_{k=1}^{K} c_k \delta_{t_k}, \quad c_k \geq 0, \quad t_k \geq 0$$

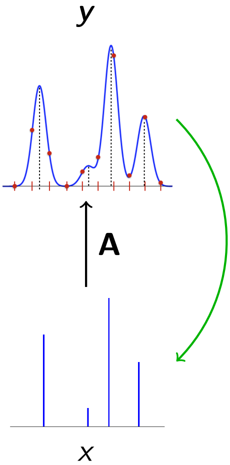
$$y(t) = \sum_{k=1}^{K} c_k h(t - t_k)$$



Sparse ℓ_0 deconvolution on a grid

$$\min_{\boldsymbol{c}\in\mathbb{R}^K} \frac{1}{2}\left\|\boldsymbol{y}-\mathbf{A}\boldsymbol{c}\right\|_2^2+\lambda\left\|\boldsymbol{c}\right\|_0$$

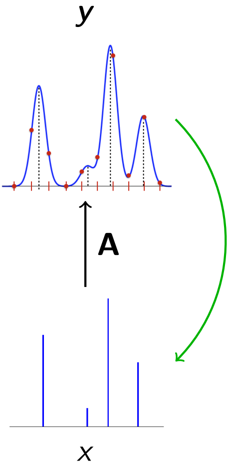
$$\boldsymbol{y} = y(\tau_i), \quad \tau_i = i\Delta/N \longrightarrow \tilde{x}_i$$



Sparse ℓ_1 deconvolution on a grid (LASSO)

$$\min_{\boldsymbol{c}\in\mathbb{R}^K} \frac{1}{2}\|\boldsymbol{y} - \mathbf{A}\boldsymbol{c}\|_2^2 + \lambda \|\boldsymbol{c}\|_1$$

$$\boldsymbol{y} = y(\tau_i), \quad \tau_i = i\Delta/N \longrightarrow \tilde{x}_i$$



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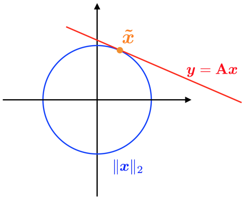
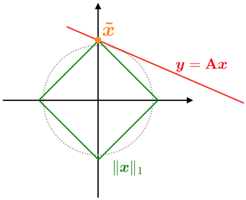
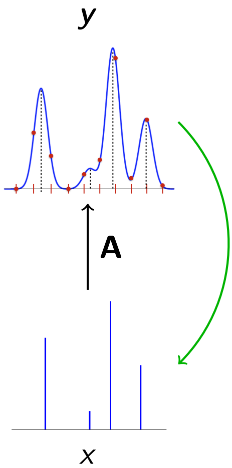
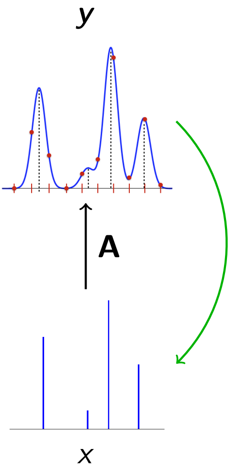






Super-resolution of 1-D impulses off-the-grid

$$x = \sum_{k=1}^{K} c_k \delta_{t_k}, \quad c_k \geq 0, \quad t_k \geq 0$$

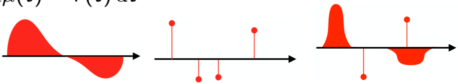


Minimization (convex regularization)

$$\arg\min_{\mu} \frac{1}{2} \|y - \mathbf{A}\mu\|^2 + \lambda \|\mu\|_{\mathrm{TV}}$$

Reference : (Candès, Fernandez-Granda, 2012)

$$\mathrm{d}\mu(t) = f(t)\,\mathrm{d}t$$



$$\|\mu\|_{\mathrm{TV}} = \int |f| \qquad \|x\|_{\mathrm{TV}} = \|\mathbf{c}\|_1$$

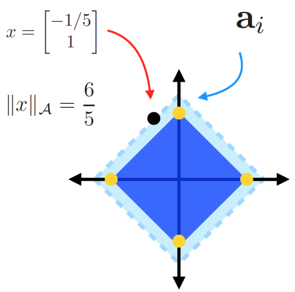
Paradigm of the atomic decomposition

$$\mathbf{x} = \sum_{k=1}^{K} c_k \mathbf{a}_i, \quad c_i \geq 0, \quad \mathbf{a}_i \in \mathcal{A}$$

Atomic norm

$$\|\mathbf{x}\|_{\mathcal{A}} = \inf \{ t > 0 : \mathbf{x} \in t\,\mathrm{conv}(\mathcal{A}) \}$$
$$= \inf \left\{ \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} : \mathbf{x} = \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} \mathbf{a} \right\}$$

(Chandrasekaran et al., 2010)



$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$$

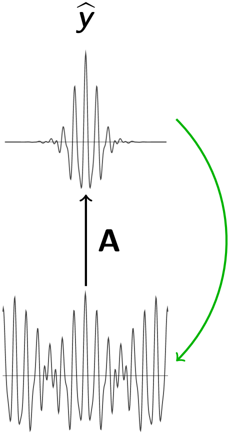
$$\|\mathbf{x}\|_{\mathcal{A}} = \|\mathbf{x}\|_1$$

Super-resolution of 1-D impulses off-the-grid

$$\widehat{\boldsymbol{x}} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_k), \quad c_k \geq 0, \quad \boldsymbol{a}(f_k) \in \mathcal{A}$$

$$\mathcal{A} = \left\{\boldsymbol{a}(f) \in \mathbb{C}^N\right\}, \quad [\boldsymbol{a}(f)]_i = \mathrm{e}^{\mathrm{j}2\pi f i}$$

$$\|\widehat{\boldsymbol{x}}\|_{\mathcal{A}} = \inf \left\{ \sum_{\boldsymbol{a}\in\mathcal{A}} c_{\boldsymbol{a}} : \widehat{\boldsymbol{x}} = \sum_{\boldsymbol{a}\in\mathcal{A}} c_{\boldsymbol{a}} \boldsymbol{a} \right\}$$



$$(\mathcal{F}x)(\omega) = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi f_k \omega}$$

Minimization (convex regularization)

$$\arg\min_{\widehat{\boldsymbol{x}}} \frac{1}{2} \|\widehat{\boldsymbol{y}} - \mathbf{A}\widehat{\boldsymbol{x}}\|^2 + \lambda \|\widehat{\boldsymbol{x}}\|_{\mathcal{A}}$$

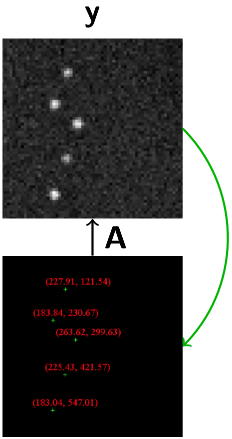
Reference : (Tang, Bhaskar, Recht et al., 2013)

Super-resolution of 2-D impulses off-the-grid

$$\widehat{\mathbf{x}} = \sum_{k=1}^{K} c_k \mathbf{a}(\boldsymbol{f}_k), \quad \boldsymbol{f}_k = (f_{k1}, f_{k2}) \in \mathbb{R}^2$$

$$\mathcal{A}_{2\mathrm{D}} = \left\{ \mathbf{a}(\boldsymbol{f}) \in \mathbb{C}^{N \times N} \right\}, \; \mathbf{a}(\boldsymbol{f}_k) = \boldsymbol{a}(f_{k1}) \otimes \boldsymbol{a}(f_{k2})$$

$$\|\widehat{\mathbf{x}}\|_{\mathcal{A}_{2\mathrm{D}}} = \inf \left\{ \sum_{\mathbf{a} \in \mathcal{A}_{2\mathrm{D}}} c_{\mathbf{a}} : \widehat{\mathbf{x}} = \sum_{\mathbf{a} \in \mathcal{A}_{2\mathrm{D}}} c_{\mathbf{a}} \mathbf{a} \right\}$$



$$x = \sum_{k=1}^{K} c_k \delta_{(t_{k1}, t_{k2})}$$

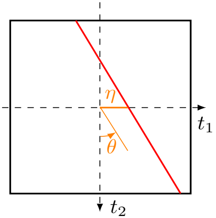
Minimization (convex regularization)

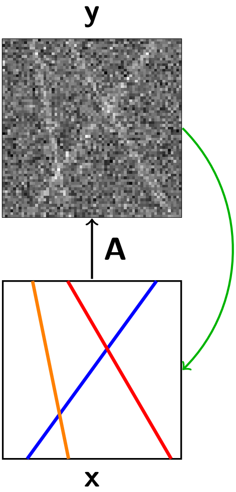
$$\arg\min_{\widehat{\mathbf{x}}} \frac{1}{2} \|\widehat{\mathbf{y}} - \mathbf{A}\widehat{\mathbf{x}}\|^2 + \lambda \|\widehat{\mathbf{x}}\|_{\mathcal{A}_{2\mathrm{D}}}$$

Reference : (Xu et al., 2013), (Chi and Chen, 2015)

Super-resolution of 2-D lines off-the-grid

$$x^\sharp(t_1,t_2) = \sum_{k=1}^{K} \alpha_k \delta\big(\cos\theta_k(t_1-\eta_k)+\sin\theta_k\, t_2\big)$$





Super-resolution of 2-D lines off-the-grid

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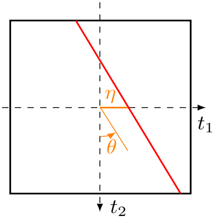
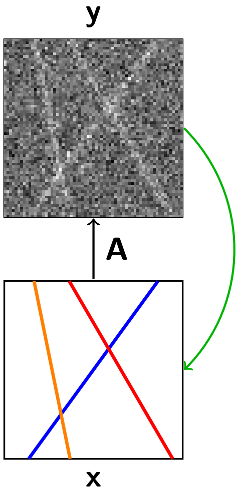
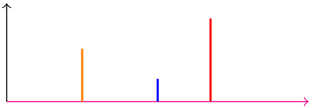


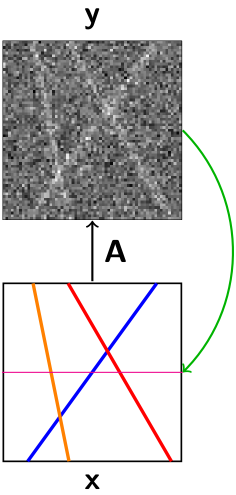


- ✓ Atomic formulation in Fourier?
- ✗ Compute atomic norm $\|\widehat{\mathbf{x}}^\sharp\|_{\mathcal{A}_{2D}}$?
- Super-resolution of 2-D lines?

Super-resolution of 2-D lines off-the-grid

$$x^{\sharp}_{t_2}(t_1) = \sum_{k=1}^{K} \alpha_k \delta\left(\cos\theta_k(\underline{t_1}-\eta_k)+\sin\theta_k\, t_2\right)$$



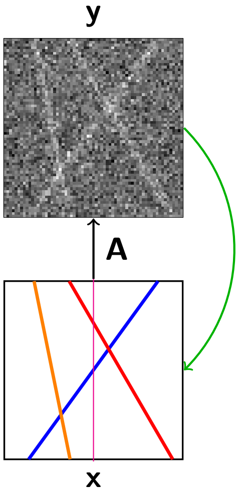


- ✓ Atomic formulation in Fourier?
- ✓ Compute atomic norm $\|\widehat{\mathbf{x}}^{\sharp}_{t_2}\|_{\mathcal{A}_{1\mathrm{D}}}$?
- ✓ Super-resolution of 2-D lines?

Super-resolution of 2-D lines off-the-grid

$$x^{\sharp}_{t_1}(t_2) = \sum_{k=1}^{K} \alpha_k \delta\big(\cos\theta_k (t_1 - \eta_k) + \sin\theta_k \, \underline{t_2}\big)$$

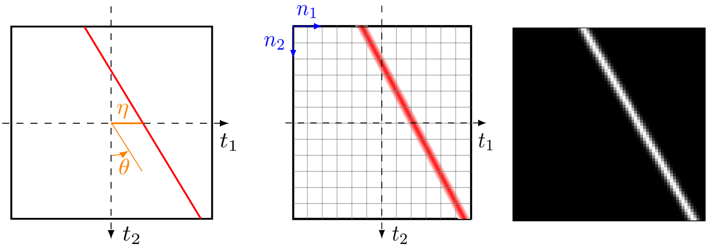
- ✓ Atomic formulation in Fourier?
- ✓ Compute atomic norm $\|\widehat{\mathbf{x}}^{\sharp}_{t_1}\|_{\mathcal{A}_{1\mathrm{D}}}$?
- ✓ Super-resolution of 2-D lines?



Modeling the blurred lines

$$x^\sharp : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^{K} \alpha_k \delta\big(\cos\theta_k (t_1 - \eta_k) + \sin\theta_k \, t_2\big)$$

$$\mathbf{b}^\sharp[n_1, n_2] = (x^\sharp * \phi)(n_1, n_2), \quad \phi(n_1, n_2) = \boldsymbol{g}[n_1]\boldsymbol{h}[n_2]$$



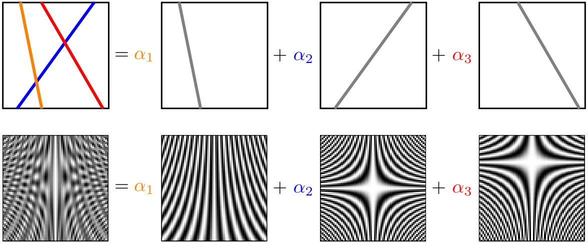
$x^\sharp$ $x^\sharp * \phi$ $\mathbf{b}^\sharp$

Modeling the blurred lines

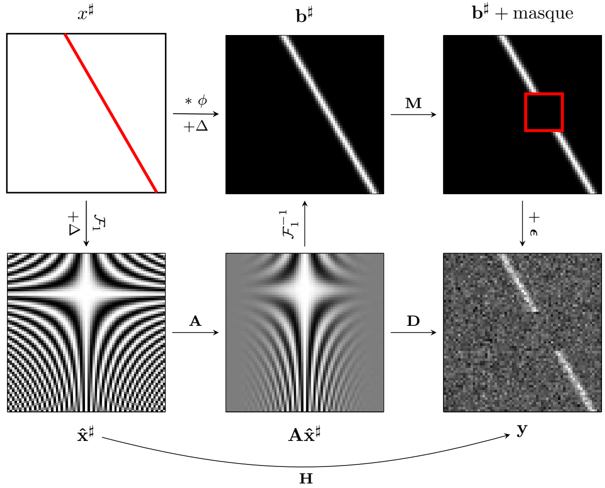
$$\widehat{\mathbf{x}}^\sharp[m,n_2] = (\mathcal{F}_1 x^\sharp)[m,n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right)m}$$

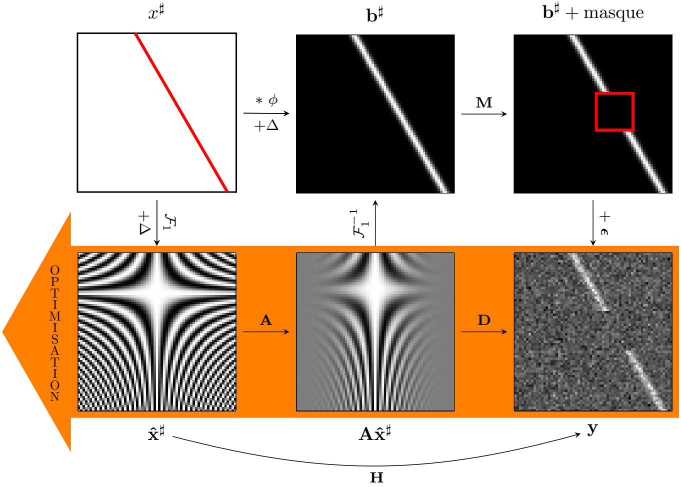
$$c_k = \frac{\alpha_k}{\cos\theta_k} \geq 0$$

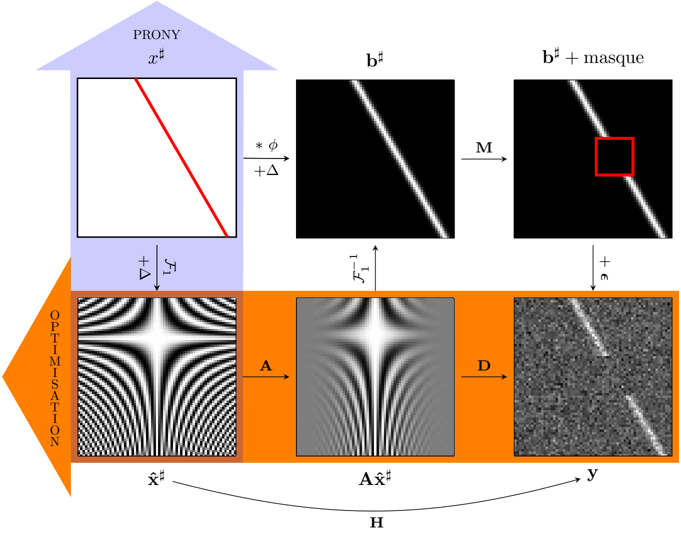
$$\widehat{\mathbf{b}}^\sharp[m,:] = (\widehat{g}[m]\widehat{\mathbf{x}}^\sharp[m,:]) * \boldsymbol{h} \rightarrow \mathbf{A}\widehat{\mathbf{x}}^\sharp = \widehat{\mathbf{b}}^\sharp$$



Reconstruction steps




$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$* \phi$
$+\Delta$
$\mathbf{M}$
$+\Delta$
$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
$+\epsilon$
OPTIMISATION
$\mathbf{A}$
$\mathbf{D}$
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
$\mathbf{H}$


PRONY
$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$*\,\phi$
$+\Delta$
$\mathbf{M}$
$+\Delta$
$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
$+\epsilon$
OPTIMISATION
$\mathbf{A}$
$\mathbf{D}$
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
$\mathbf{H}$

Atomic decomposition of columns and rows

$$\widehat{\mathbf{x}}^\sharp[\underline{m}, n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right)m}, \quad m = -M, \ldots, M$$

$$\widehat{\mathbf{x}}^\sharp[m, \underline{n_2}] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} m\right)n_2 + \frac{2\pi\eta_k m}{W}}, \quad n_2 = 0, \ldots, H-1$$

1. $l^\sharp_{n_2} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k}, \boxed{0})$ (columns of $\widehat{\mathbf{x}}$, without phase)
2. $t^\sharp_m = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \boxed{\phi_{m,k}})^\mathsf{T}$ (rows of $\widehat{\mathbf{x}}$, with phase)

$$[\boldsymbol{a}(f,\phi)]_i = \mathrm{e}^{\mathrm{j}(2\pi f i + \phi)} \in \mathcal{A}$$

Caratheodory theorem

Theorem (Caratheodory, 1907)

A vector $\mathbf{z} = (z_{N-1}^*, \ldots, z_1^*, z_0, z_1, \ldots, z_{N-1})$, *with* $z_0 \in \mathbb{R}$, *is a positive combination of* $K \leqslant N$ *atoms* $\mathbf{a}(f_k, \boxed{0})$ *if and only if* $\mathbf{T}_N(\mathbf{z}_+) \succcurlyeq 0$ *and is of rank* K, *where* $\mathbf{z}_+ = (z_0, \ldots, z_{N-1})$ *and*

$$\mathbf{T}_N : \mathbf{z}_+ = (z_0, \ldots, z_{N-1}) \mapsto \begin{pmatrix} z_0 & z_1^* & \cdots & z_{N-1}^* \\ z_1 & z_0 & \cdots & z_{N-2}^* \\ \vdots & \vdots & \ddots & \vdots \\ z_{N-1} & z_{N-2} & \cdots & z_0 \end{pmatrix}.$$

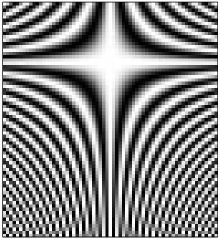
Moreover, this decomposition is unique if $K < N$.

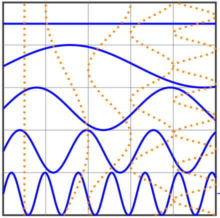
Atomic decomposition of one line ($K = 1$)

$$\widehat{\mathbf{x}}^\sharp[m, n_2] = c_1 \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_1}{W} n_2 + \frac{\eta_1}{W}\right)m} = c_1 \mathrm{e}^{\mathrm{j}2\pi((f_1 - f_0)n_2 + f_0)m}$$

with $f_0 = -\eta_1/W$ and $f_1 = (\tan\theta_1 - \eta_1)/W$.

1. $\boldsymbol{l}^\sharp_{n_2} = c_1 \boldsymbol{a}(f_{n_2,1}, 0)$ (one atom without phase)
2. $\boldsymbol{t}^\sharp_m = c_1 \boldsymbol{a}(f_{m,1}, \phi_{m,1})^\mathsf{T}$ (one atom with phase)





Atomic characterization of one line ($K = 1$)

Characterization of one sampled line in Fourier

A matrix $\widehat{\mathbf{x}}$ is of the form $\widehat{\mathbf{x}}[m,n] = c_1 \mathrm{e}^{\mathrm{j}2\pi((f_1 - f_0)n + f_0)m}$ if and only if the columns $\boldsymbol{l}_n$ and rows $\boldsymbol{t}_m$ of $\widehat{\mathbf{x}}$ are such that $\mathbf{T}_M(\boldsymbol{l}_n) \succcurlyeq 0$ and of rank 1, $\mathbf{P}_1(\boldsymbol{t}_m)$ is of rank 1 and $\widehat{\mathbf{x}}[0,n] = \widehat{\mathbf{x}}[0,0]$ for all m and n.

$$\underbrace{\begin{pmatrix} z_0 & z_1^* & \cdots & z_{N-1}^* \\ z_1 & z_0 & \cdots & z_{N-2}^* \\ \vdots & \vdots & \ddots & \vdots \\ z_{N-1} & z_{N-2} & \cdots & z_0 \end{pmatrix}}_{=\mathbf{T}_N(\boldsymbol{z})} \quad \underbrace{\begin{pmatrix} z_K & z_{K-1} & \cdots & z_0 \\ z_{K+1} & z_K & \cdots & z_1 \\ \vdots & \vdots & \ddots & \vdots \\ z_{N-1} & z_{N-2} & \cdots & z_{N-K-1} \end{pmatrix}}_{=\mathbf{P}_K(\boldsymbol{z})}.$$

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Convex relaxation: since $\mathbf{T}_M(\boldsymbol{l}_{n_2}) = \mathbf{V}_{n_2} \mathrm{diag}(c_1, \ldots, c_K) \mathbf{V}_{n_2}^*$ with the Vandermonde matrix $\mathbf{V}_{n_2} = (\boldsymbol{a}(f_{n_2,1}), \ldots, \boldsymbol{a}(f_{n_2,K}))$, we get:

$$\|\mathbf{T}_M(\boldsymbol{l}_{n_2})\|_* \propto \|\boldsymbol{l}_{n_2}\|_{\mathcal{A}}$$

Atomic norms computation

1. $\boldsymbol{l}^\sharp_{n_2} = \sum_{k=1}^K c_k \boldsymbol{a}(f_{n_2,k}, \boxed{0}) \Leftrightarrow \mathbf{T}_{M+1}(\boldsymbol{l}^\sharp_{n_2}) \succcurlyeq 0$ + of rank K

Atomic norm without phase (Caratheodory, 1907)

Since $K < M$ the decomposition is unique then

$$\|\boldsymbol{l}^\sharp_{n_2}\|_{\mathcal{A}} = \sum_{k=1}^K c_k = \widehat{\mathbf{x}}^\sharp[0, n_2] = c^\star$$

2. $\boldsymbol{t}^\sharp_m = \sum_{k=1}^K c_k \boldsymbol{a}(f_{m,k}, \boxed{\phi_{m,k}})^\mathsf{T}$

Atomic norm with phase (Tang et al., 2013)

$$\|\boldsymbol{t}^\sharp_m\|_{\mathcal{A}} = \inf_{\boldsymbol{q}\in\mathbb{C}^H, t\in\mathbb{R}} \left\{ \frac{1}{2}\mathrm{Tr}(\mathbf{T}_N(\boldsymbol{q})) + \frac{1}{2}t : \begin{pmatrix} \mathbf{T}_N(\boldsymbol{q}) & \boldsymbol{t}^\sharp_m \\ {\boldsymbol{t}^\sharp_m}^* & t \end{pmatrix} \succcurlyeq 0 \right\}$$

Atomic norms computation

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Atomic norm with phase

$$\|\boldsymbol{t}^\sharp_m\|_{\mathcal{A}} = \min_{\boldsymbol{q} \in \mathbb{C}^H} \left\{ q_0 : \underbrace{\begin{pmatrix} \mathbf{T}_N(\boldsymbol{q}) & \boldsymbol{t}^\sharp_m \\ \boldsymbol{t}^{\sharp\,*}_m & q_0 \end{pmatrix}}_{\mathbf{T}'_N(\boldsymbol{t}^\sharp_m, \boldsymbol{q})} \succcurlyeq 0 \right\} \equiv \underbrace{\mathrm{SDP}(\boldsymbol{t}^\sharp_m)}_{\boldsymbol{q}_m[0]} \leqslant c^\star$$

Convex optimization problem

Proposition (Convex minimization)

$$\tilde{\mathbf{x}} \in \underset{\widehat{\mathbf{x}},\mathbf{q} \in \mathcal{X}\times\mathcal{Q}}{\arg\min} \frac{1}{2}\|\mathbf{A}\widehat{\mathbf{x}} - \widehat{\mathbf{y}}\|^2 \ ,$$

$$\text{under constraints} \quad \begin{cases} \forall n_2 = 0, \ldots, H-1, \ \forall m = 1, \ldots, M \ , \\ \widehat{\mathbf{x}}[0, n_2] = \widehat{\mathbf{x}}[0,0] \leqslant c \ , \\ \mathbf{q}[m, 0] \leqslant c \ , \\ \mathbf{T}'_H(\widehat{\mathbf{x}}[m,:], \mathbf{q}[m,:]) \succcurlyeq 0 \ , \\ \mathbf{T}_{M+1}(\widehat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 \ . \end{cases}$$

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Proposition (Convex minimization)

$$\tilde{\mathbf{x}} \in \underset{\widehat{\mathbf{x}},\mathbf{q} \in \mathcal{X} \times \mathcal{Q}}{\arg\min} \frac{1}{2} \|\mathbf{A}\widehat{\mathbf{x}} - \widehat{\mathbf{y}}\|^2 ,$$

$$\text{under constraints} \quad \begin{cases} \forall n_2 = 0, \dots, H-1, \ \forall m = 1, \dots, M , \\ \widehat{\mathbf{x}}[0, n_2] = \widehat{\mathbf{x}}[0,0] \leqslant c , \\ \mathbf{q}[m, 0] \leqslant c , \\ \mathbf{T}'_H(\widehat{\mathbf{x}}[m, :], \mathbf{q}[m, :]) \succcurlyeq 0 , \\ \mathbf{T}_{M+1}(\widehat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 . \end{cases}$$

(Chambolle et Pock, 2010)

$$\tilde{\mathbf{X}} = \underset{\mathbf{X} \in \mathcal{H}}{\arg\min} \left\{ F(\mathbf{X}) + G(\mathbf{X}) + \sum_{i=0}^{Q-1} H_i(\mathrm{L}_i(\mathbf{X})) \right\}$$

Convex optimization problem

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Convex optimization problem

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under constraints

$$\begin{cases} \forall n_2 = 0, \dots, H-1,\ \forall m = 1, \dots, M , \\ \widehat{\mathbf{x}}[0, n_2] = \widehat{\mathbf{x}}[0, 0] \leqslant c , \\ \mathbf{q}[m, 0] \leqslant c , \\ \mathbf{T}'_H(\widehat{\mathbf{x}}[m, :], \mathbf{q}[m, :]) \succcurlyeq 0 , \\ \mathbf{T}_{M+1}(\widehat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 . \end{cases}$$

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Convex optimization problem

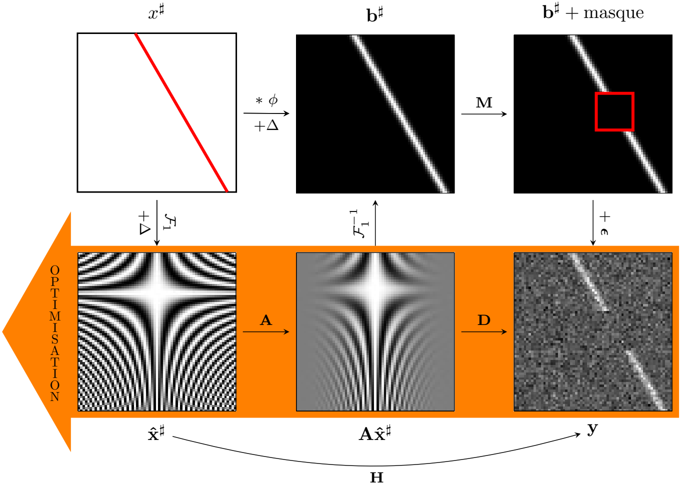
Proposition (Convex minimization)

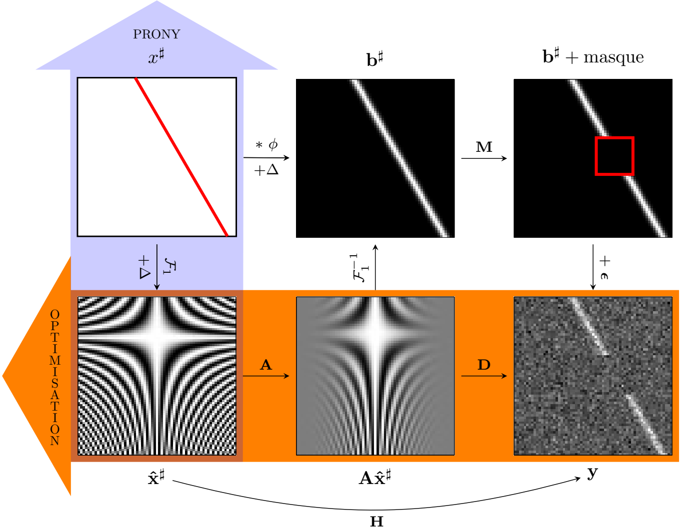
$$\tilde{\mathbf{x}} \in \underset{\widehat{\mathbf{x}},\mathbf{q} \in \mathcal{X} \times \mathcal{Q}}{\arg\min} \frac{1}{2} \|\mathbf{A}\widehat{\mathbf{x}} - \widehat{\mathbf{y}}\|^2 ,$$

$$\text{under constraints} \quad \begin{cases} \forall n_2 = 0, \ldots, H-1, \ \forall m = 1, \ldots, M , \\ \widehat{\mathbf{x}}[0, n_2] = \widehat{\mathbf{x}}[0, 0] \leqslant c , \\ \mathbf{q}[m, 0] \leqslant c , \\ \mathbf{T}'_H(\widehat{\mathbf{x}}[m, :], \mathbf{q}[m, :]) \succcurlyeq 0 , \\ \mathbf{T}_{M+1}(\widehat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 . \end{cases}$$

(Chambolle and Pock, 2010)

$$\tilde{\mathbf{X}} = \underset{\mathbf{X} \in \mathcal{H}}{\arg\min} \left\{ F(\mathbf{X}) + G(\mathbf{X}) + \sum_{i=0}^{Q-1} H_i(\mathrm{L}_i(\mathbf{X})) \right\}$$


$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$* \phi$
$+\Delta$
M
$\mathcal{F}_1$
$+\Delta$
$\mathcal{F}_1^{-1}$
$+\epsilon$
OPTIMISATION
A
D
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
H


PRONY
$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$* \phi$
$+\Delta$
$\mathbf{M}$
$\mathcal{F}_1$
$+\Delta$
$\mathcal{F}_1^{-1}$
$+\epsilon$
OPTIMISATION
$\mathbf{A}$
$\mathbf{D}$
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
$\mathbf{H}$

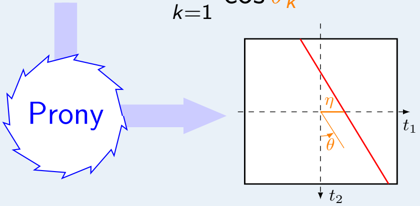
Super-resolution of lines in two steps

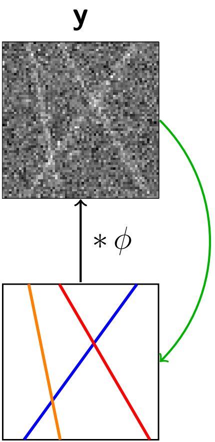
Convex optimization $\mathbf{y} \to \widehat{\mathbf{x}}$

$$\arg\min_{\widehat{\mathbf{x}}} \frac{1}{2} \|\widehat{\mathbf{y}} - \mathbf{A}\widehat{\mathbf{x}}\|^2 + \lambda \sum_n \|\widehat{x}_n\|_{\mathcal{A}_{1\mathrm{D}}}$$

Spectral estimation $\widehat{\mathbf{x}} \to \{\alpha_k, \theta_k, \eta_k\}_k$

$$\widehat{\mathbf{x}}^\sharp[m, n_2] = \sum_{k=1}^{K} \frac{\alpha_k}{\cos\theta_k} \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right)m}$$



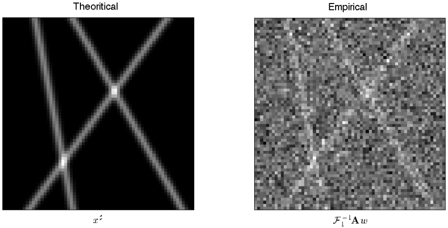


$$\{\alpha_k, \theta_k, \eta_k\}_{k=1}^{K} \to x^\sharp$$

Numerical experiments

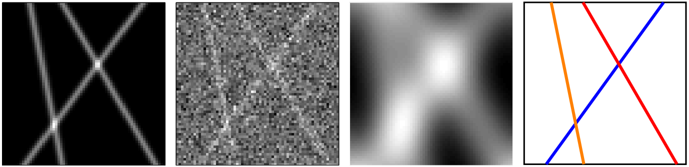
- Denoising lines:

image000



Numerical experiments

- Denoising and deconvolution



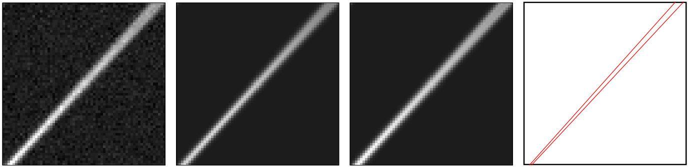
Exp. 1 | Exp. 2 | Exp 3. | Détection

Table: Relative errors of the line parameters estimation

	Experiment 1	**Experiment 2**	**Experiment 3**
Δ_θ/θ	$(10^{-7}, 3.10^{-6}, 7.10^{-7})$	$(10^{-2}, 6.10^{-2}, 9.10^{-2})$	$(6.10^{-7}, 9.10^{-5}, 8.10^{-6})$
Δ_α/α	$(10^{-7}, 10^{-7}, 10^{-7})$	$(10^{-2}, 9.10^{-2}, 2.10^{-1})$	$(4.10^{-5}, 2.10^{-5}, 2.10^{-5})$
Δ_η	$(4.10^{-6}, 7.10^{-6}, 7.10^{-6})$	$(5.10^{-2}, 4.10^{-2}, 3.10^{-2})$	$(5.10^{-5}, 10^{-4}, 3.10^{-4})$

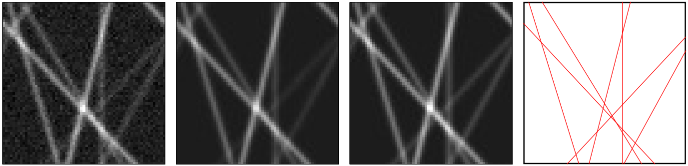
Numerical experiments

- Closed lines



Noisy | Noisy | Ground truth | Detection

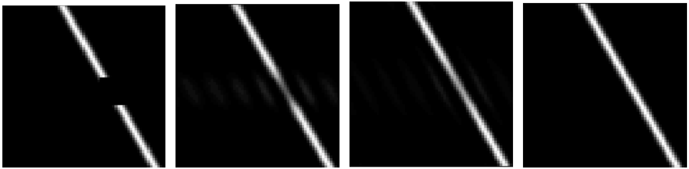
- Multiple lines



Noisy | Noisy | Ground trutht | Detection

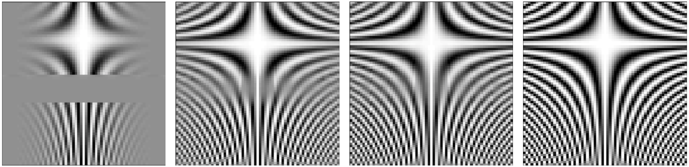
Numerical experiments

- Spatial inpainting



Masking | iter $= 2000$ | iter $= 10000$ | iter $\to \infty$

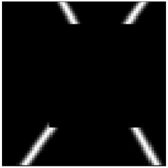
- Inpainting in Fourier



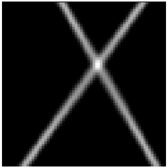
Masquage | iter $= 2000$ | iter $= 10000$ | iter $\to \infty$

Numerical experiments

- Inpainting with big mask

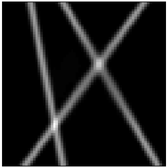


Masquage



Inpainting

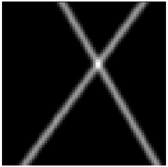
Masquage



Inpainting

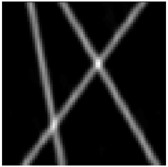
- Inpainting with random mask

Masquage



Inpainting

Masquage



Inpainting

Conclusion

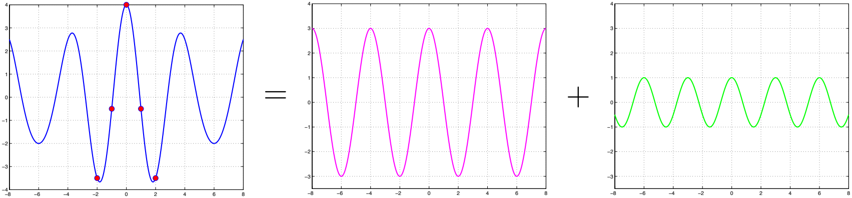
- ✓ New method for the super-resolution of 2-D lines, with dedicated 1-D atomic norms penalities enforcing sparsity
- ✓ Penalize in both directions can lead to the exact solution
- ✗ No theoretical guarantees about separation conditions and statistical analysis for recovering the exact solution
- ✓ Solving the convex optimization problem by a primal-dual splitting algorithm
- ✗ Slow convergence : for each iteration perform SVD onto all the Toeplitz matrices made from rows and columns
- ✓ Extraction of the line parameters combining spectral estimation on rows/columns achieves subpixel accuracy

Questions?

Thank you for your attention

Frequencies extraction from columns and rows

1. $\boldsymbol{l}^{\sharp}_{n_2} = \sum_{k=1}^{K} c_k \mathbf{a}(f_{n_2,k}, 0)$ (columns of $\widehat{\mathbf{x}}$, without phase)
2. $\boldsymbol{t}^{\sharp}_{m} = \sum_{k=1}^{K} c_k \mathbf{a}(f_{m,k}, \phi_{m,k})^{\mathsf{T}}$ (rows de $\widehat{\mathbf{x}}$, with phase)



$x(t) = x_1(t) + x_2(t)$ $\quad$ $x_1(t) = \boxed{3}\exp\left(\mathrm{j}2\pi\boxed{\frac{1}{4}}t\right)$ $\quad$ $x_2(t) = \boxed{1}\exp\left(\mathrm{j}2\pi\boxed{\frac{1}{3}}t\right)$

$\Rightarrow$ spectral method estimation (Prony, ESPRIT, MUSIC, Matrix Pencil...)

Prony method

$$x_m = \sum_{k=1}^{K} \rho_k \underbrace{\left(e^{-j\omega_k}\right)}_{z_k}^{m}, \quad \rho_k \in \mathbb{C},\ \omega_k \in [-\pi, \pi],\ m = -M, \ldots, M$$

Annihilating filter : $H(z) = \prod_{k=1}^{K} (z - \overline{z_k}) = \sum_{k=0}^{K} h_k z^k$

$$\sum_{j=0}^{K} h_j x_{m-j} = \sum_{j=0}^{K} h_j \left(\sum_{k=1}^{K} \rho_k z_k^{m-j} \right) = \sum_{k=1}^{K} \rho_k z_k^{m} \underbrace{\left(\sum_{j=0}^{K} h_j z_k^{-j} \right)}_{H(\overline{z_k})=0} = 0$$

Prony method : annihilating polynomial

$$x_m = \sum_{k=1}^{K} \rho_k \underbrace{\left(e^{-j\omega_k}\right)}_{z_k}^{m}, \quad \rho_k \in \mathbb{C},\ \omega_k \in [-\pi, \pi],\ m = -M, \ldots, M$$

Annihilating filter : $H(z) = \prod_{k=1}^{K} (z - \overline{z_k}) = \sum_{k=0}^{K} h_k z^k$

- $\sum_{j=0}^{K} h_j x_{m-j} = 0,\ \forall m = -M+K, \ldots, M \Leftrightarrow \boldsymbol{x} * \boldsymbol{h} = \boldsymbol{0}$
- $\begin{pmatrix} x_{-M+K} & \cdots & x_{-M} \\ \vdots & \ddots & \vdots \\ x_M & \cdots & x_{M-K} \end{pmatrix} \begin{pmatrix} h_0 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Leftrightarrow \mathbf{T}_K \boldsymbol{h} = \boldsymbol{0}$

Prony method : frequencies estimation

$$x_m = \sum_{k=1}^{K} \rho_k \underbrace{\left(e^{-j\omega_k}\right)}_{z_k}^{m}, \quad \rho_k \in \mathbb{C},\ \omega_k \in [-\pi, \pi],\ m = -M, \ldots, M$$

Annihilating filter: $H(z) = \prod_{k=1}^{K}(z - \overline{z_k}) = \sum_{k=0}^{K} h_k z^k$

- $\boldsymbol{h}$ = sing. vec. for $\lambda = 0$ of

$$\mathbf{T}_K = \begin{pmatrix} x_{-M+K} & \cdots & x_{-M} \\ \vdots & \ddots & \vdots \\ x_M & \cdots & x_{M-K} \end{pmatrix}$$

- $\overline{z_k}$ = roots of the polynomial $H(z)$, puis $\omega_k = \arg(\overline{z_k})$

Prony method : amplitudes estimation

- $x_m = \sum_{k=1}^{K} \rho_k \left(e^{-j\omega_k}\right)^m, \forall m = -M, \ldots, M$
- $\begin{pmatrix} e^{jM\omega_1} & \cdots & e^{jM\omega_K} \\ \vdots & \ddots & \vdots \\ e^{-jM\omega_1} & \cdots & e^{-jM\omega_K} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_K \end{pmatrix} = \begin{pmatrix} x_{-M} \\ \vdots \\ x_M \end{pmatrix} \Leftrightarrow \mathbf{U}\boldsymbol{\rho} = \boldsymbol{x}$

Least-square method :

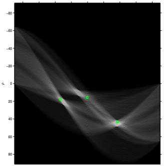
$$\mathbf{U}^{\mathrm{H}}\mathbf{U}\boldsymbol{\rho} = \mathbf{U}^{\mathrm{H}}\boldsymbol{x} \Longleftrightarrow \boldsymbol{\rho} = (\mathbf{U}^{\mathrm{H}}\mathbf{U})^{-1}\mathbf{U}^{\mathrm{H}}\boldsymbol{x}$$

Expériences numériques

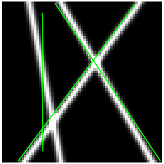
Avec la transformée de Hough

- Espace de Hough pour la détection de lignes :

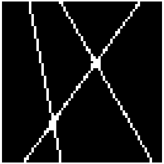
Squelettisation



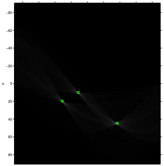
Détection pics



Reconstruction



Squelettisation



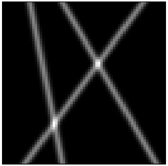
Détection pics

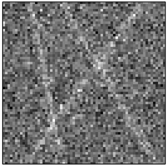
Reconstruction

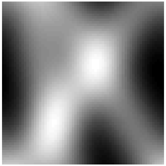
Expériences numériques

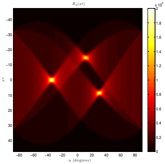
Avec la transformée de Radon

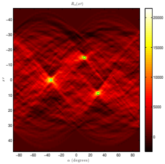
- Espace de Radon pour la détection de lignes :

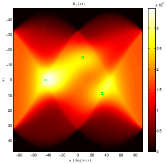

Exp. 1


Exp. 2


Exp 3.

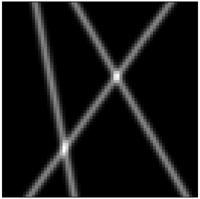

Détection pics

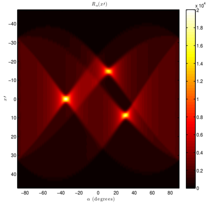

Détection pics

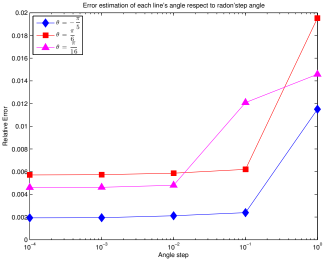

Détection pics

Expériences numériques
Avec la transformée de Radon

- Limite de ces transformées discrètes attachées à la grille :







Exp. 1

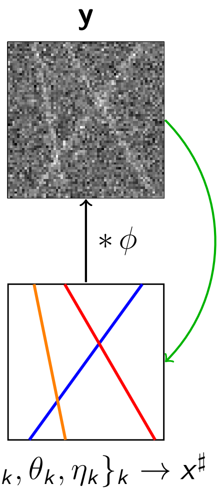
Atomic formulation in 2-D

Method in 2 steps:
denoising/deconvolution + extraction

1. Convex minimization ⇒ slowness
2. Spectral estimation ⇒ robustness

New method in the parameters space

Conditional gradient method (Frank-Wolfe)[1] : $\mathbf{y} \to \{\alpha_k, \theta_k, \eta_k\}_k$



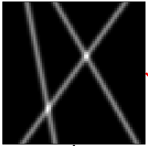
[1] (Boyd, 2015) The Alternating Descent Conditional Gradient Method

Atomic formulation in 2-D

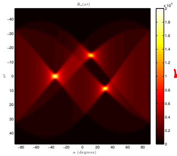
$$\mathbf{y} = \int \psi(\Theta)\, \mathrm{d}\mu^\sharp = \Phi\mu^\sharp$$

New method in the parameters space

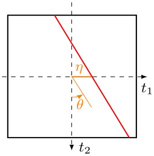
Conditional gradient method (Frank-Wolfe): $\mathbf{y} \to \{\alpha_k, \theta_k, \eta_k\}_k$



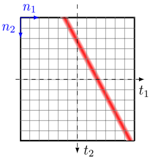
$\mathcal{R}\mathbf{y}$



$$\min_{\mu \geq 0} \ell(\Phi\mu - \mathbf{y}), \ |\mathrm{supp}(\mu)| \leq K$$



$\Theta = (\theta, \eta)$



$\psi(\Theta)$

$$\mu_0 \approx \mu^\sharp = \sum_{k=1}^{K} \alpha_k \delta_{\Theta_k}$$