Convex Super-Resolution Detection of Lines in Images

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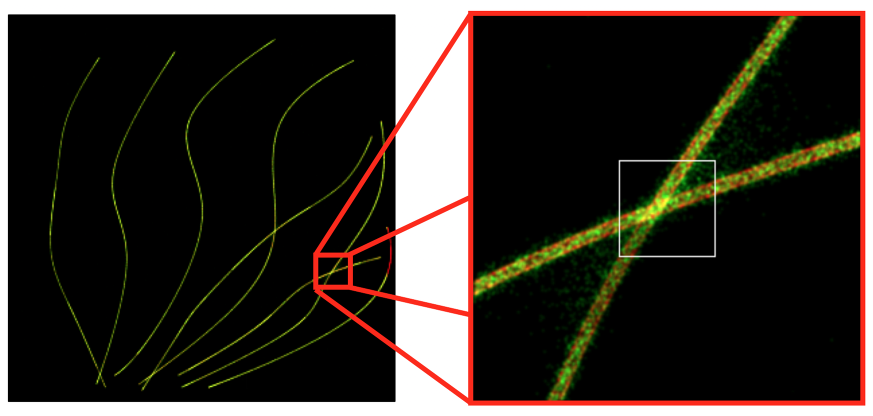
Objectives

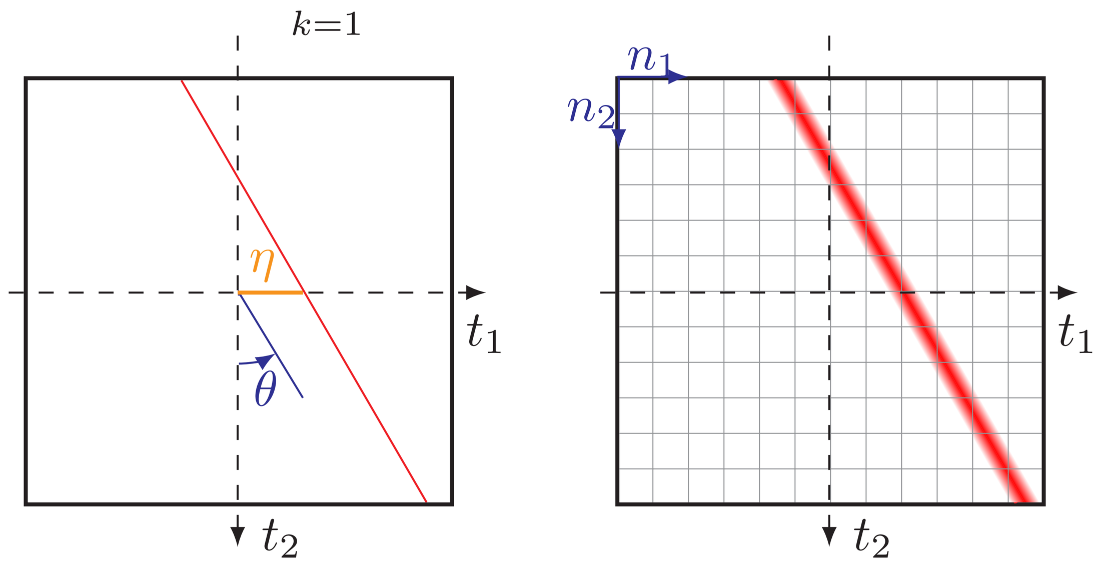
We present a new convex formulation for the problem of recovering lines in degraded images. Following the recent paradigm of super-resolution, we formulate a dedicated atomic norm penalty and solve this optimization problem by a primal–dual algorithm. Then, a spectral estimation method recovers the line parameters, with subpixel accuracy.



Model of Noisy Blurred Lines

A sum of K perfect lines of infinite length, with angle $\theta_k \in (-\pi/2, \pi/2]$, amplitude $\alpha_k > 0$, and offset $\eta_k \in \mathbb{R}$, is defined as the distribution

$$x^\sharp(t_1, t_2) = \sum_{k=1}^{K} \alpha_k \delta(\cos\theta_k (t_1 - \eta_k) + \sin\theta_k t_2).$$



The image observed $b^\sharp$ of size $W \times H$ is obtained by the convolution of $x^\sharp$ with a blur function ϕ, following by a sampling with unit step Δ: $b^\sharp[n_1, n_2] = (x^\sharp * \phi)(n_1, n_2)$. The point spread function ϕ is separable, that is $x^\sharp * \phi$ can be obtained by a first horizontal convolution $u^\sharp = x^\sharp * \varphi_1$, where φ_1 is W-periodic and bandlimited, that is its Fourier coefficients $\hat{g}[m]$ are zero for $|m| \geq (W-1)/1 = M+1$, so $\hat{u}^\sharp[m, n_2] = \hat{g}[m]\hat{x}^\sharp[m, n_2]$; and then a second vertical convolution with φ_2, such as the discrete filter $h[n] = (\varphi_2 * \mathrm{sinc})[n]$ has compact support, gives $\hat{b}^\sharp[m, :] = \hat{u}^\sharp[m, :] * h = \hat{g}[m]\hat{x}[m, :] * h$, hence $\mathbf{A}\hat{x}^\sharp = \hat{b}^\sharp$

$$\hat{x}^\sharp[m, n_2] = \sum_{k=1}^{K} \frac{\alpha_k}{\cos\theta_k} e^{j2\pi(\tan\theta_k n_2 - \eta_k)m/W}.$$

Atomic Norm Framework

Let $z \in \mathbb{C}^N$ be a vector such as $z = \sum_{k=1}^{K} c_k a(\omega_k)$ with $c_k \in \mathbb{C}$ and *atoms* $a(\omega) \in \mathbb{C}^N$ continuously indexed in a dictionary $\mathcal{A}$ by a parameter ω in a compact set Ω. The atomic norm, which enforces sparsity with respect this set $\mathcal{A}$, is defined as

$$\|z\|_{\mathcal{A}} = \inf_{c'_k, \omega'_k} \left\{ \sum_k |c'_k| : z = \sum_k c'_k a(\omega'_k) \right\}.$$

Consider the dictionary

$$\mathcal{A} = \{a(f, \phi) \in \mathbb{C}^{|I|}, f \in [0,1], \phi \in [0, 2\pi)\},$$

in which the *atoms* are the vectors of components $[a(f, \phi)]_i = e^{j(2\pi f i + \phi)}, i \in I$, and simply $[a(f)]_i = e^{j2\pi f i}, i \in I$, if $\phi = 0$. The atomic norm writes:

$$\|z\|_{\mathcal{A}} = \inf_{c'_k > 0, f'_k, \phi'_k} \left\{ \sum_k c'_k : z = \sum_k c'_k a(f'_k, \phi'_k) \right\}.$$

Theorem 1 [Caratheodory]. Let $z = (z_n)_{n=-N+1}^{N-1}$ be a vector with Hermitian symmetry $z_{-n} = z_n^*$. z is a positive combination of $K \leqslant N+1$ atoms $a(f_k)$ if and only if $\mathbf{T}_N(z_+) \succcurlyeq 0$ and of rank K, where $z_+ = (z_0, \ldots, z_{N-1})$ and $\mathbf{T}_N$ is the Toeplitz operator

$$\mathbf{T}_N(z_+) = \begin{pmatrix} z_0 & z_1^* & \cdots & z_{N-1}^* \\ z_1 & z_0 & \cdots & z_{N-2}^* \\ \vdots & \vdots & \ddots & \vdots \\ z_{N-1} & z_{N-2} & \cdots & z_0 \end{pmatrix}.$$

Moreover, this decomposition is unique, if $K \leqslant N$.

Proposition 1. The atomic norm $\|z\|_{\mathcal{A}}$ can be characterized by this semidefinite program SDP(z) [2]:

$$\|z\|_{\mathcal{A}} = \min_{q \in \mathbb{C}^N} \left\{ q_0 : \mathbf{T}'_N(z, q) = \begin{pmatrix} \mathbf{T}_N(q) & z \\ z^* & q_0 \end{pmatrix} \succcurlyeq 0 \right\}.$$

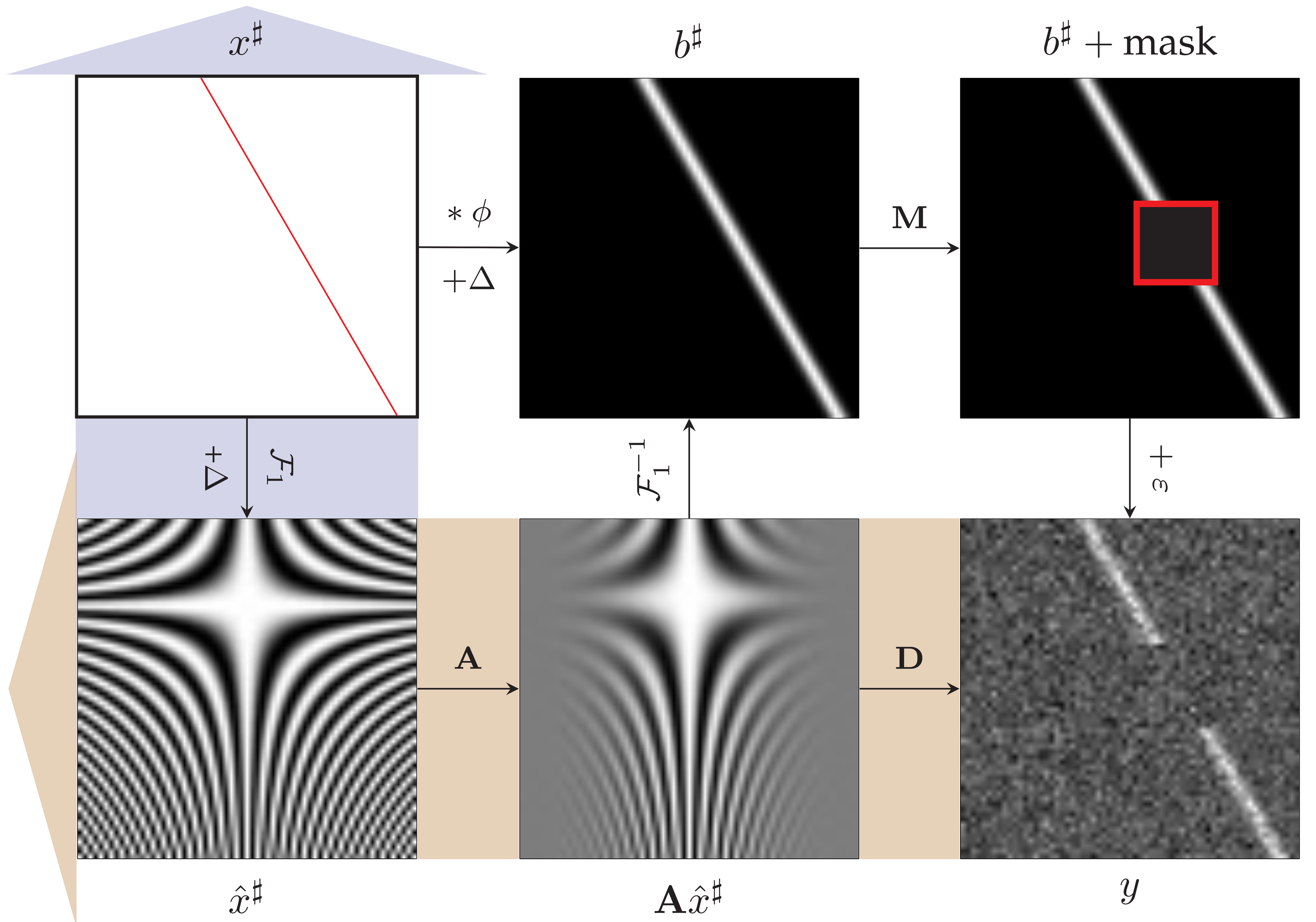
- $l^\sharp_{n_2} = \hat{x}^\sharp[:, n_2] = \sum_{k=1}^{K} c_k a(f_{n_2,k})$
- $t^\sharp_m = \hat{x}^\sharp[m, :] = \sum_{k=1}^{K} c_k a(f_{m,k}, \phi_{m,k})^T$ with

amplitude $c_k = \dfrac{\alpha_k}{\cos\theta_k}$, phase $\phi_{m,k} = -\dfrac{2\pi\eta_k m}{W}$,

frequency $f_{n_2,k} = \dfrac{\tan\theta_k n_2 - \eta_k}{W}$, $f_{m,k} = \dfrac{\tan\theta_k m}{W}$.

- $\|l^\sharp_{n_2}\|_{\mathcal{A}} = \sum_{k=1}^{K} c_k = \hat{x}^\sharp[0, n_2]$ by **Theorem 1**.
- $\|t^\sharp_m\|_{\mathcal{A}} = \mathrm{SDP}(t^\sharp_m) \leqslant \sum_{k=1}^{K} c_k$ by **Proposition 1**.

Super-Resolution and Regularization of Lines



(Observation operator)
$\mathbf{H} = \mathbf{M}\mathcal{F}_1^{-1}\mathbf{A}$

A : blur operator
$\mathcal{F}_1$: horiz. Fourier operator
M : inpainting operator

$$\tilde{x} \in \underset{\hat{x}, q \in \mathcal{X} \times \mathcal{Q}}{\arg\min} \frac{1}{2}\|\mathbf{H}\hat{x} - y\|_F^2,$$

$$\forall n_2, m \begin{cases} \hat{x}[0, n_2] = \hat{x}[0, 0] \leqslant c, \\ q[m, 0] \leqslant c, \\ \mathbf{T}'_{H_S}(\hat{x}[m, :], q[m, :]) \succcurlyeq 0, \\ \mathbf{T}_{M+1}(\hat{x}[:, n_2]) \succcurlyeq 0, \end{cases}$$

The problem can be rewritten in this way:

$$\tilde{X} = \underset{X \in \mathcal{H}}{\arg\min} \left\{ F(X) + G(X) + \sum_{i=0}^{N-1} H_i(L_i(X)) \right\}$$

with $F(X) = \frac{1}{2}\|\mathbf{H}\hat{x} - y\|_F^2$, $X = (\hat{x}, q)$, ∇F a β–Lipschitz gradient, a proximable indicator $G = \iota_{\mathcal{B}}$ where $\mathcal{B}$ are the two first boundary constraints, and $N = M + 1 + H_S$ linear composite terms, where $H_i = \iota_{\mathcal{C}}$ with $\mathcal{C}$ the cone of semidefinite positive matrices, and $L_i \in \{L^{(1)}_m, L^{(2)}_{n_2}\}$, defined by $L^{(1)}_m(X) = \mathbf{T}'_{H_S}(\hat{x}[m, :], q[m, :])$ and $L^{(2)}_{n_2}(X) = \mathbf{T}_{M+1}(\hat{x}[:, n_2])$. **L** denotes the concatenation of the L_i operators.

Let $\tau > 0$ and $\sigma > 0$ such that $\frac{1}{\tau} - \sigma\|\mathbf{L}\|^2 \geqslant \frac{\beta}{2}$.

Algorithm: Primal–dual splitting method [Condat]

Input: The blurred and noisy data image y
Output: $\tilde{x}$ solution of the optimization problem

1: Initialize all primal and dual variables to zero
2: **for** $n = 1$ **to** Number of iterations **do**
3: $\quad X_{n+1} = \mathrm{prox}_{\tau G}(X_n - \tau\nabla F(X_n) - \tau\sum_i L_i^* \xi_{i,n})$,
4: $\quad$ **for** $i = 0$ **to** $N-1$ **do**
5: $\quad\quad \xi_{i,n+1} = \mathrm{prox}_{\sigma H_i^*}(\xi_{i,n} + \sigma L_i(2X_{n+1} - X_n))$,
6: $\quad$ **end for**
7: **end for**

Spectral Estimation by a Prony-like Method

Let be $d_k \in \mathbb{C}$, $f_k \in [-1/2, 1/2)$, $\zeta_k = e^{j2\pi f_k}$ and

$$z_i = \sum_{k=1}^{K} d_k \left(e^{j2\pi f_k}\right)^i, \quad \forall i = 0, \ldots, |I| - 1,$$

The **annihilating polynomial filter** is defined by:
$H(\zeta) = \prod_{l=1}^{K}(\zeta - \zeta_l) = \sum_{l=0}^{K} h_l \zeta^{K-l}$ with $h_0 = 1$,

$$\sum_{l=0}^{K} h_l z_{r-l} = \sum_{k=1}^{K} d_k \zeta_k^{r-K} \underbrace{\left(\sum_{l=0}^{K} h_l \zeta_k^{K-l}\right)}_{H(\zeta_k)=0} = 0.$$

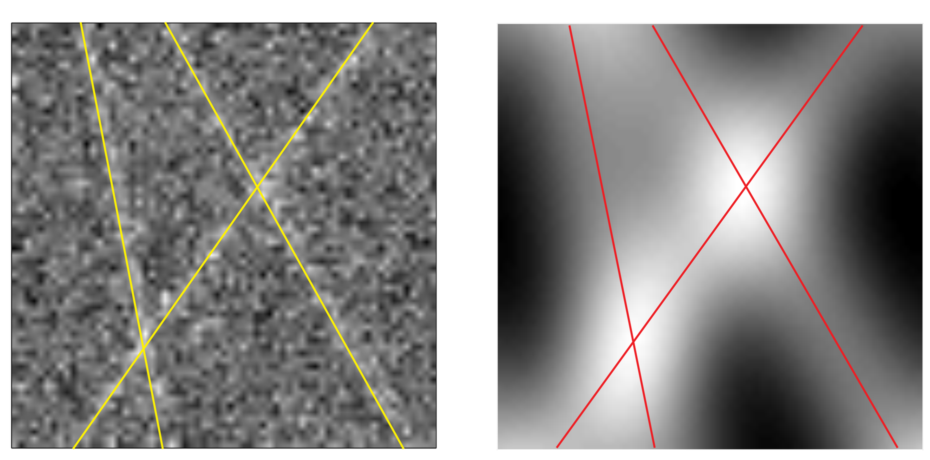
$$\mathbf{P}_K(z)h = \begin{pmatrix} z_K & \cdots & z_0 \\ \vdots & \ddots & \vdots \\ z_{|I|-1} & \cdots & z_{|I|-K-1} \end{pmatrix} \begin{pmatrix} h_0 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

❶ From $\mathbf{P}_K(z)$, compute h by a SVD. Form H whose roots give access to the frequencies f_k.
❷ Since $z = \mathbf{U}d$ with $\mathbf{U} = (a(f_1), \cdots, a(f_K))$, find amplitudes by LS: $d = (\mathbf{U^H U})^{-1}\mathbf{U^H}z$.

Procedure for retrieving the line parameters

❶ For each column $\tilde{x}[m, :]$ compute $\{\tilde{f}_{m,k}\}_k$ by ❶
❷ For each column $\tilde{x}[m, :]$ compute $\{\tilde{d}_{m,k}\}_k$ by ❷
❸ $\{f_{m,k}\}_m = \{\frac{\tan\theta_k m}{W}\}_m$ lin. regression $\to \{\tilde{\theta}_k\}$
❹ $\tilde{\alpha}_{m,k} = |\tilde{d}_{m,k}| \cos(\tilde{\theta}_k)$ and $\{\alpha_k\}_k = \mathbb{E}[\{\tilde{\alpha}_{m,k}\}_m]$
❺ $\tilde{d}_{m,k}/|\tilde{d}_{m,k}| = (e^{-j2\pi\frac{\eta_k}{W}})^m \to \{\eta_k\}_k$ by ❶

This procedure enables to estimate the line parameters from the solution $\tilde{x}$ of the optimization problem:



References

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[2] B. N. Bhaskar *et al.*, Atomic norm denoising with applications to line spectral estimation, IEEE Transactions on signal processing, 2013.
[3] G. Tang *et al.*, Compressed sensing off the grid, IEEE Transactions on information theory, 2013.