A Convex Approach to Super-Resolution and Regularization of Lines in Images

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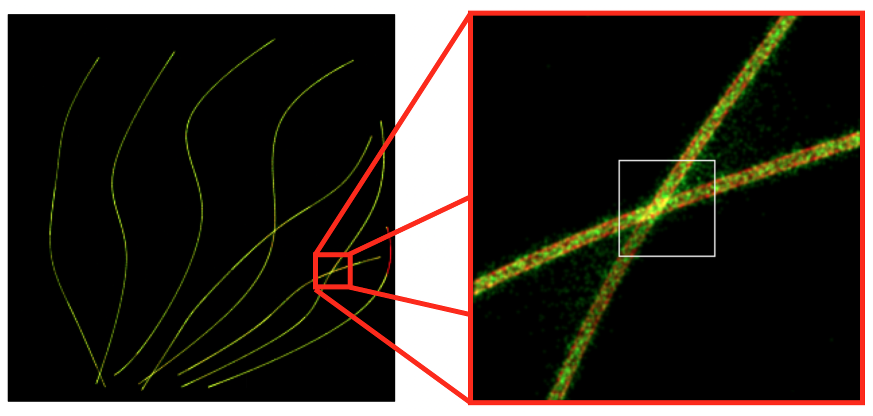
Objectives

We present a new convex formulation for the problem of recovering lines in degraded images. Following the recent paradigm of super-resolution, we formulate a dedicated atomic norm penalty and solve this optimization problem by a primal–dual algorithm. Then, a spectral estimation method recovers the line parameters, with subpixel accuracy.



Atomic Norm Framework

Let $\boldsymbol{z} \in \mathbb{C}^N$ be a vector represented as a linear positive finite combinaison of sampled complex exponentials $[\boldsymbol{a}(f,\phi)]_i = \mathrm{e}^{\mathrm{j}(2\pi f i + \phi)}$, that is

$$\boldsymbol{z} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_k, \phi_k),$$

whose parameters f_k and ϕ_k are continuously indexed in a dictionary of *atoms*:

$$\mathcal{A} = \{\boldsymbol{a}(f,\phi) \in \mathbb{C}^N, f \in [0,1], \phi \in [0, 2\pi)\}.$$

The atomic norm, which enforces sparsity with respect to the set $\mathcal{A}$, is defined as

$$\|\boldsymbol{z}\|_{\mathcal{A}} = \inf_{c'_k > 0, f'_k, \phi'_k} \left\{ \sum_k c'_k : \boldsymbol{z} = \sum_k c'_k \boldsymbol{a}(f'_k, \phi'_k) \right\}.$$

Theorem 1 [Caratheodory]. Let $\boldsymbol{z} = (z_n)_{n=-N+1}^{N-1}$ be a vector with Hermitian symmetry $z_{-n} = z_n^*$. $\boldsymbol{z}$ is a positive combination of $K \leqslant N$ atoms $\boldsymbol{a}(f_k, 0)$ if and only if $\mathbf{T}_N(\boldsymbol{z}_+) \succcurlyeq 0$ and of rank K, where $\boldsymbol{z}_+ = (z_0, \dots, z_{N-1})$ and $\mathbf{T}_N$ is the Toeplitz operator

$$\mathbf{T}_N(\boldsymbol{z}_+) = \begin{pmatrix} z_0 & z_1^* & \cdots & z_{N-1}^* \\ z_1 & z_0 & \cdots & z_{N-2}^* \\ \vdots & \vdots & \ddots & \vdots \\ z_{N-1} & z_{N-2} & \cdots & z_0 \end{pmatrix}.$$

Moreover, this decomposition is unique, if $K < N$.

Proposition 1. The atomic norm $\|\boldsymbol{z}\|_{\mathcal{A}}$ can be characterized by this semidefinite program SDP($\boldsymbol{z}$):

$$\|\boldsymbol{z}\|_{\mathcal{A}} = \min_{\boldsymbol{q} \in \mathbb{C}^N} \left\{ q_0 : \mathbf{T}'_N(\boldsymbol{z}, \boldsymbol{q}) = \begin{pmatrix} \mathbf{T}_N(\boldsymbol{q}) & \boldsymbol{z} \\ \boldsymbol{z}^* & q_0 \end{pmatrix} \succcurlyeq 0 \right\}.$$

Primal-Dual Algorithm

Let $\tau > 0$ and $\sigma > 0$ such that $\frac{1}{\tau} - \sigma\|\mathbf{L}\|^2 > \frac{\beta}{2}$.

Input: The blurred and noisy data image $\mathbf{y}$
Output: $\tilde{\mathbf{x}}$ solution of the optimization problem (1)

1. Initialize all primal and dual variables to zero
2. **for** $n = 1$ **to** Number of iterations **do**
3. $\quad \mathbf{X}_{n+1} = \mathrm{prox}_{\tau G}(\mathbf{X}_n - \tau \nabla F(\mathbf{X}_n) - \tau \sum_{i=1}^{Q} \mathrm{L}_i^* \boldsymbol{\xi}_{i,n})$
4. $\quad$ **for** $i = 0$ **to** $Q-1$ **do**
5. $\quad\quad \boldsymbol{\xi}_{i,n+1} = \mathrm{prox}_{\sigma H_i^*}(\boldsymbol{\xi}_{i,n} + \sigma \mathrm{L}_i(2\mathbf{X}_{n+1} - \mathbf{X}_n))$
6. $\quad$ **end for**
7. **end for**

References

[1] K. Polisano *et al.*, Convex super-resolution detection of lines in images, IEEE EUSIPCO, 2016.

[2] K. Polisano *et al.*, A convex approach to super-resolution and regularization of lines in images, *preprint*, 2018.

Super-Resolution and Regularization of Lines

The Model. A sum of K perfect lines of infinite length, with angle $\theta_k \in (-\pi/2, \pi/2]$, amplitude $\alpha_k > 0$ and offset $\eta_k \in \mathbb{R}$, is defined as the distribution

$$x^\sharp(t_1, t_2) = \sum_{k=1}^{K} \alpha_k \delta(\cos\theta_k\,(t_1 - \eta_k) + \sin\theta_k\, t_2).$$

The blurred image $\mathbf{b}^\sharp$ of size $W \times H$ is obtained by the convolution of $x^\sharp$ with a separated blur function ϕ, following by a sampling with unit step Δ:

$$\mathbf{b}^\sharp[n_1, n_2] = (x^\sharp * \phi)(n_1, n_2).$$

We denote by $\widehat{\mathbf{x}}^\sharp = \mathcal{F}_1 x^\sharp$ (resp. $\widehat{\mathbf{b}}^\sharp$) the horizontal Fourier transform of $x^\sharp$ (resp. $\mathbf{b}^\sharp$), whose expression after sampling is the following Hermitian matrix:

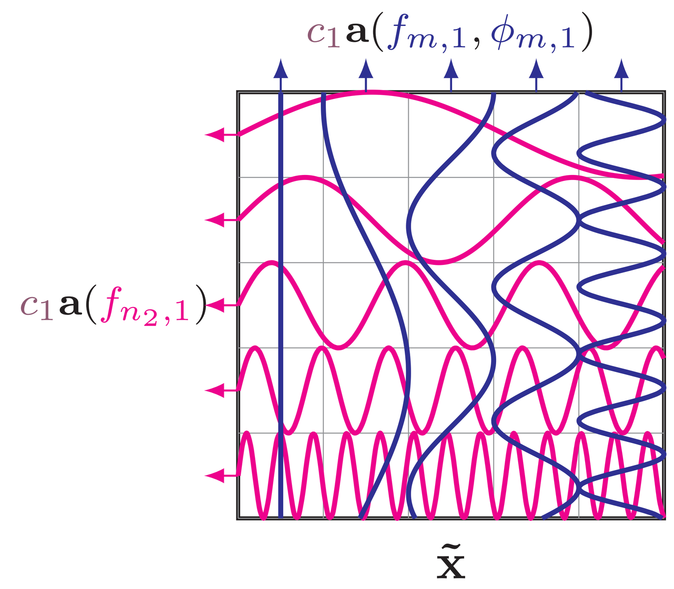
$$\widehat{\mathbf{x}}^\sharp[m, n_2] = \sum_{k=1}^{K} \frac{\alpha_k}{\cos\theta_k} \mathrm{e}^{\mathrm{j}2\pi(\tan\theta_k\, n_2 - \eta_k)m/W},$$

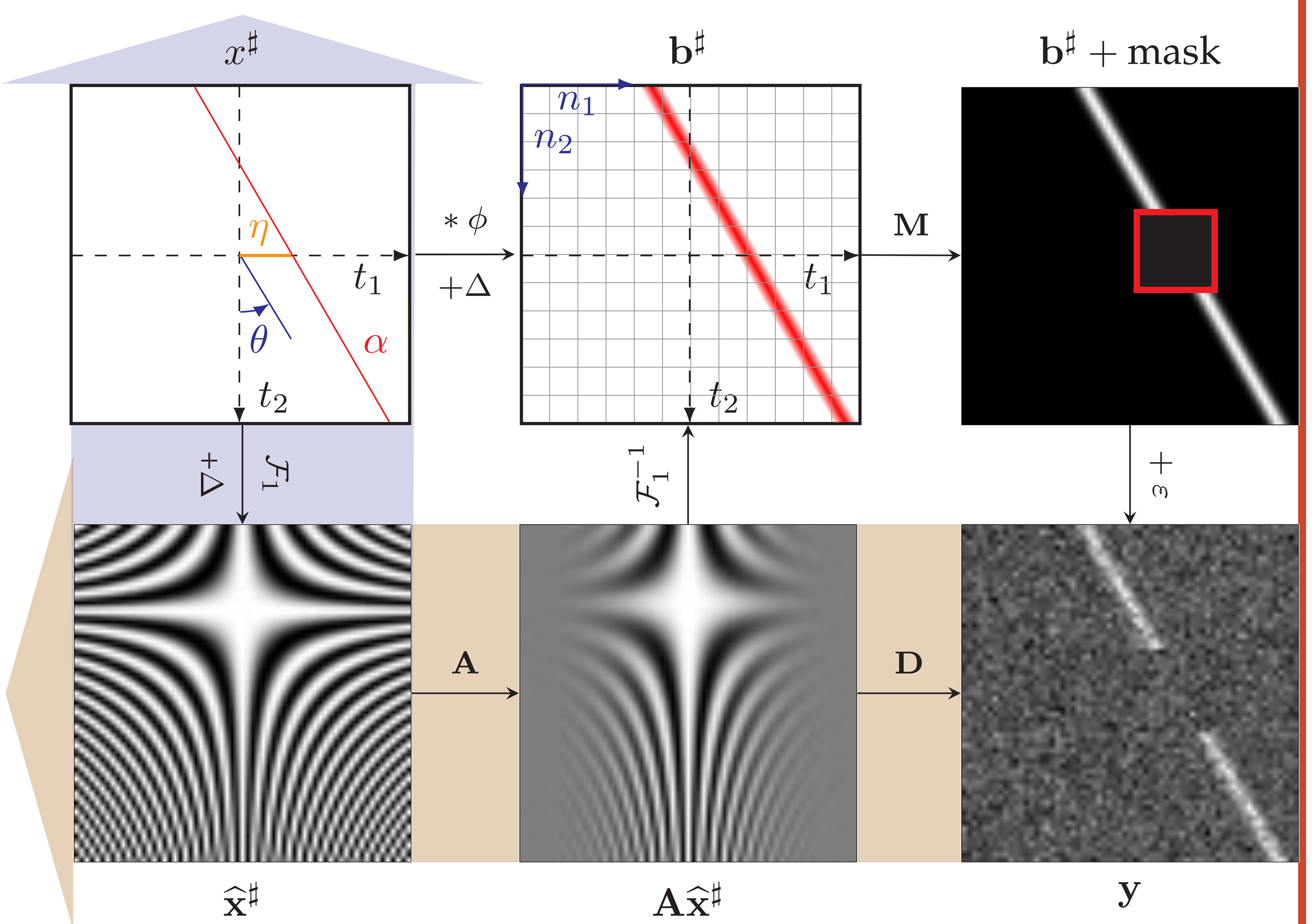
for $m = -M, \dots, M$, $n_2 = 0, \dots, H$, $M = (W-1)/2$. In the discrete paradigm, the blur function ϕ with suitable assumptions corresponds in Fourier to a linear operator $\mathbf{A}$, leading to the inverse problem:

$$\mathbf{A}\widehat{\mathbf{x}}^\sharp = \widehat{\mathbf{b}}^\sharp.$$

The observed image $\mathbf{y}$ is possibly affected by an inpainting mask $\mathbf{M}$ and some white noise $\boldsymbol{\epsilon}$, that is

$$\mathbf{y} = \mathbf{M}\mathcal{F}_1^{-1}\mathbf{A}\widehat{\mathbf{x}}^\sharp + \boldsymbol{\epsilon} = \mathbf{H}\widehat{\mathbf{x}}^\sharp + \boldsymbol{\epsilon}.$$





$$\tilde{\mathbf{x}} \in \underset{\widehat{\mathbf{x}}, \mathbf{q} \in \mathcal{X} \times \mathcal{Q}}{\arg\min} \frac{1}{2}\|\mathbf{H}\widehat{\mathbf{x}} - \mathbf{y}\|_{\mathrm{F}}^2$$
$$\begin{cases} \forall n_2 = 0, \dots, H-1 \\ \forall m = 0, \dots, M \\ \widehat{\mathbf{x}}[0, n_2] = \widehat{\mathbf{x}}[0,0] \leqslant c \\ \mathbf{q}[m, 0] \leqslant c \\ \mathbf{T}'_{H_S}(\widehat{\mathbf{x}}[m,:], \mathbf{q}[m,:]) \succcurlyeq 0 \\ \mathbf{T}_{M+1}(\widehat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 \end{cases}$$

- $\boldsymbol{l}^\sharp_{n_2} = \widehat{\mathbf{x}}^\sharp[:, n_2] = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k})$

$$c_k = \frac{\alpha_k}{\cos\theta_k}, \quad f_{n_2,k} = \frac{\tan\theta_k\, n_2 - \eta_k}{W}.$$

- $\boldsymbol{t}^\sharp_m = \widehat{\mathbf{x}}^\sharp[m, :] = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^\mathsf{T}$

$$f_{m,k} = \frac{\tan\theta_k\, m}{W}, \quad \phi_{m,k} = -\frac{2\pi\eta_k m}{W}, \quad d_{m,k} = c_k \phi_{m,k}.$$

Lines regularization. Minimizing these atomic norms simultaneously enables to enforce sparsity decomposition on rows and columns of the solution:

- $\|\boldsymbol{l}^\sharp_{n_2}\|_{\mathcal{A}} = \sum_{k=1}^{K} c_k = \widehat{\mathbf{x}}^\sharp[0, n_2]$ (**Theorem 1**)
- $\|\boldsymbol{t}^\sharp_m\|_{\mathcal{A}} = \mathrm{SDP}(\boldsymbol{t}^\sharp_m) \leqslant \sum_{k=1}^{K} c_k$ (**Proposition 1**)

Resolution. The optimization problem is rewritten:

$$\tilde{\mathbf{X}} = \underset{\mathbf{X} \in \mathcal{H}}{\arg\min} \left\{ F(\mathbf{X}) + G(\mathbf{X}) + \sum_{i=0}^{Q-1} H_i \circ \mathrm{L}_i(\mathbf{X}) \right\}, \quad (1)$$

with $F(\mathbf{X}) = \frac{1}{2}\|\mathbf{H}\widehat{\mathbf{x}} - \mathbf{y}\|_{\mathrm{F}}^2$, $\mathbf{X} = (\widehat{\mathbf{x}}, \mathbf{q})$, ∇F a β–Lipschitz gradient, a proximable indicator $G = \iota_{\mathcal{B}}$ where $\mathcal{B}$ are the two first boundary constraints, and $Q = M + 1 + H$ linear composite terms, where $H_i = \iota_{\mathcal{C}}$ with $\mathcal{C}$ the cone of semidefinite positive matrices and $\mathrm{L}_i \in \{\mathrm{L}_m^{(1)}, \mathrm{L}_{n_2}^{(2)}\}$, $\mathrm{L}_m^{(1)}(\mathbf{X}) = \mathbf{T}'_{H_S}(\widehat{\mathbf{x}}[m,:], \mathbf{q}[m,:])$ and $\mathrm{L}_{n_2}^{(2)}(\mathbf{X}) = \mathbf{T}_{M+1}(\widehat{\mathbf{x}}[:, n_2])$. $\mathbf{L}$ denotes the concatenation of the L_i operators in the next algorithm.

Spectral Estimation by a Prony-like Method

Let be $d_k \in \mathbb{C}$, $f_k \in [-1/2, 1/2)$, $\zeta_k = \mathrm{e}^{\mathrm{j}2\pi f_k}$ and

$$z_i = \sum_{k=1}^{K} d_k \left(\mathrm{e}^{\mathrm{j}2\pi f_k}\right)^i, \quad \forall i = 0, \dots, N-1.$$

The **annihilating polynomial filter** is defined by: $H(\zeta) = \prod_{l=1}^{K}(\zeta - \zeta_l) = \sum_{l=0}^{K} h_l \zeta^{K-l}$ with $h_0 = 1$,

$$\sum_{l=0}^{K} h_l z_{r-l} = \sum_{k=1}^{K} d_k \zeta_k^{r-K} \underbrace{\left(\sum_{l=0}^{K} h_l \zeta_k^{K-l}\right)}_{H(\zeta_k)=0} = 0.$$

$$\mathbf{P}_K(\boldsymbol{z})\boldsymbol{h} = \begin{pmatrix} z_K & \cdots & z_0 \\ \vdots & \ddots & \vdots \\ z_{N-1} & \cdots & z_{N-K-1} \end{pmatrix} \begin{pmatrix} h_0 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

❶ From $\mathbf{P}_K(\boldsymbol{z})$, compute $\boldsymbol{h}$ by a SVD. Form H whose roots give access to the frequencies f_k.
❷ Since $\boldsymbol{z} = \mathbf{U}\boldsymbol{d}$ with $\mathbf{U} = (\boldsymbol{a}(f_1), \cdots, \boldsymbol{a}(f_K))$, find amplitudes by LS: $\boldsymbol{d} = (\mathbf{U}^{\mathbf{H}}\mathbf{U})^{-1}\mathbf{U}^{\mathbf{H}}\boldsymbol{z}$.

Procedure for retrieving the line parameters

❶ For each column $\tilde{\mathbf{x}}[m,:]$ compute $\{\tilde{f}_{m,k}\}_k$ by ❶
❷ For each column $\tilde{\mathbf{x}}[m,:]$ compute $\{\tilde{d}_{m,k}\}_k$ by ❷
❸ $\{f_{m,k}\}_m = \{\frac{\tan\theta_k m}{W}\}_m$ lin. regression $\rightarrow \{\tilde{\theta}_k\}$
❹ $\tilde{\alpha}_{m,k} = |\tilde{d}_{m,k}|\cos(\tilde{\theta}_k) \rightarrow \{\tilde{\alpha}_k\}_k = \mathbb{E}[\{\tilde{\alpha}_{m,k}\}_m]$
❺ $d_{m,k}/|d_{m,k}| = (\mathrm{e}^{-\mathrm{j}2\pi\frac{\eta_k}{W}})^m \rightarrow \{\tilde{\eta}_k\}_k$ by ❶

This procedure enables to super-resolve the line parameters from the solution $\tilde{\mathbf{x}}$ of the problem (1).