Locally Anisotropic Fractional Brownian Field

{ Kévin Polisano, Marianne Clausel, Valérie Perrier and Laurent Condat }

Introduction

We introduce a new class of Gaussian fields, called *Locally Anisotropic Fractional Brownian Fields*, with prescribed local orientation and Hurst index at any point. These fields are a local version of a specific class of anisotropic self-similar Gaussian fields with stationary increments.

Global Anisotropic Fields

Let $0 < H < 1$. The *fractional Brownian field* B^H with Hurst index H, is the unique centered Gaussian field with stationary increments :

- $z \in \mathbb{R}^2, B^H(\cdot + z) - B^H(z) \overset{\mathcal{L}}{=} B^H(\cdot) - B^H(0)$,
- self-similar : $\forall \lambda \in \mathbb{R}^\star, B^H(\lambda \cdot) \overset{\mathcal{L}}{=} \lambda^H B^H(\cdot)$,
- isotropic : $\forall$ rotation R in $\mathbb{R}^2$, $B^H \circ R \overset{\mathcal{L}}{=} B^H$,

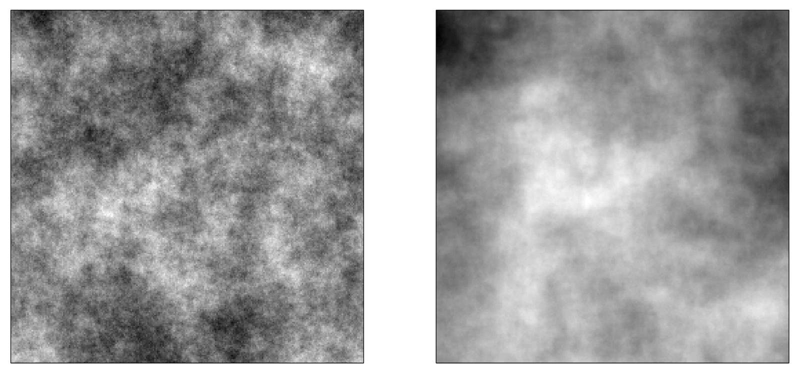


Figure 3: H=0.2 (on the left) and H=0.7 (on the right)

In its harmonizable representation which is

$$B^H(x) = \int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi} - 1}{\|\xi\|^{H+1}} d\widehat{W}(\xi),$$

we can introduce anisotropy by restricted the frequency components of the spectral density $f^{1/2}(\xi) = c(\arg\xi)\|\xi\|^{-H-1}$ into a cone in taking

$$B^H{}_{\alpha_0,\alpha}(x) = \int_{\mathbb{R}^2} (e^{ix\cdot\xi} - 1) \frac{\mathbb{1}_{[-\alpha,\alpha]}(\arg\xi - \alpha_0)}{\|\xi\|^{H+1}} d\widehat{W}(\xi)$$

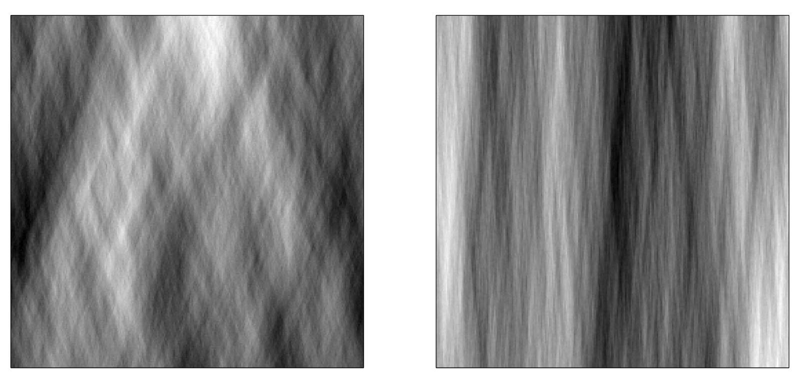


Figure 4: Elementary field $\alpha_0 = 0$, $H = 0.5$, $\alpha \in \{\frac{\pi}{6}, \frac{\pi}{24}\}$

Tangent field

The random field X is locally asymptotically self-similar of order $H \in (0,1)$ if for any $h \in \mathbb{R}^2$ the random field

$$\frac{X(x_0 + \rho h) - X(x_0)}{\rho^H},$$

admits a non-trivial limit in law Y_{x_0} as $\rho \to 0$. The field Y_{x_0} is then called the tangent field of X at x_0. Roughly speaking, the random field X admits the tangent field Y_{x_0} at a given point x_0 if it **behaves locally as** Y_{x_0} when $x \to x_0$.

We can prove that the LAFBF X admits a tangent field Y_{x_0} at any point $x_0 \in \mathbb{R}^2$ defined as:

$$Y_{x_0}(x) = \int_{\mathbb{R}^2} (e^{ix\xi} - 1) \frac{\mathbb{1}_{[-\alpha,\alpha]}(\arg\xi - \alpha_0(x_0))}{\|\xi\|^{H+1}} d\widehat{W}(\xi)$$

Kriging method

The basic idea of kriging is to predict the value of a random field $Z(\cdot)$ at an unobserved location s_0 by computing a weighted average of the known sampled values of the field in the neighborhood $\mathcal{V}(s_0)$ of the point s_0.

$$\hat{Z}(s_i) = a + \sum_{i \in \mathcal{V}(s_0)} \lambda_i Z(s_i) = a + \lambda^t \mathbf{Z}$$

The aim is to find the weight λ_i for which :

- $\mathbb{E}[\hat{Z}(s_0) - Z(s_0)] = 0$
- The variance $\mathrm{Var}[\hat{Z}(s_0) - Z(s_0)]$ is minimal

The Best Linear Unbiased Estimator (BLUE) is obtained with $\hat{\lambda} = \mathbb{E}[\mathbf{Z}\mathbf{Z}^t]^{-1}\mathrm{Cov}(\mathbf{Z}, Z(s_0))$

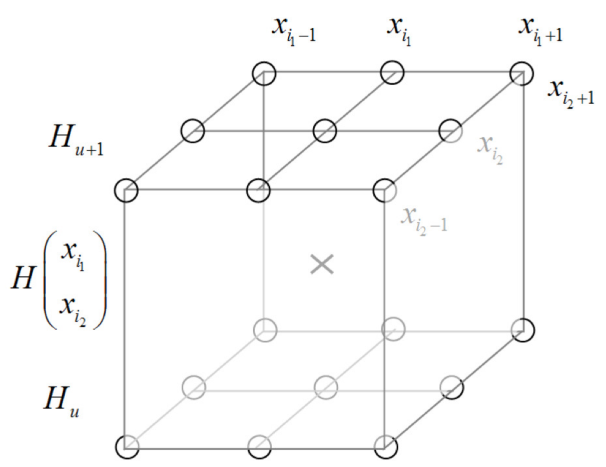
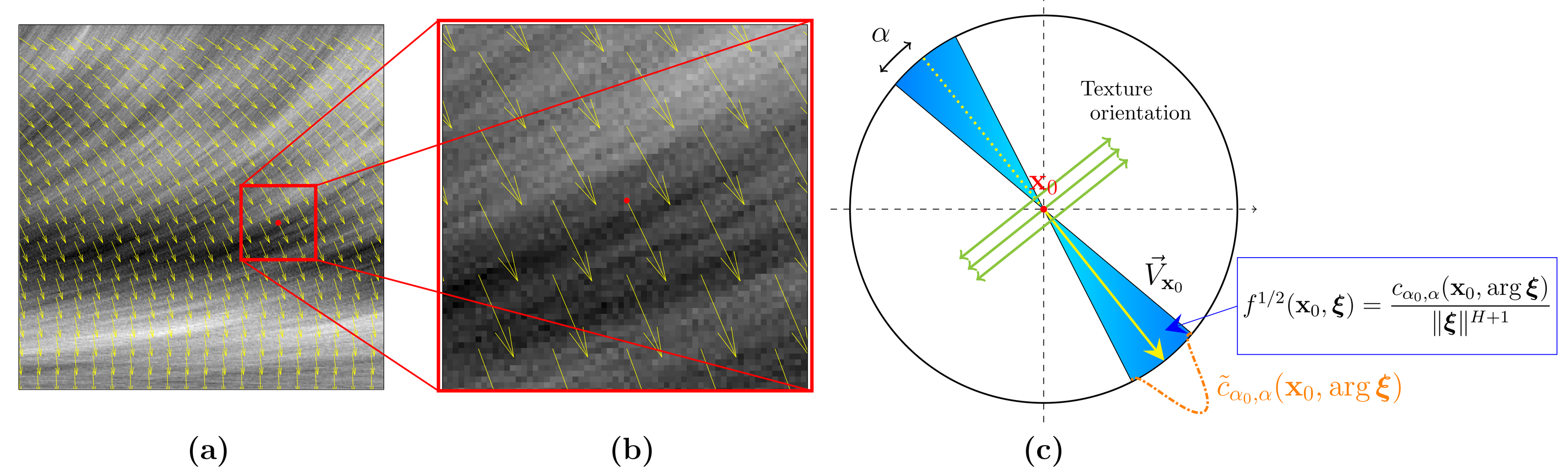
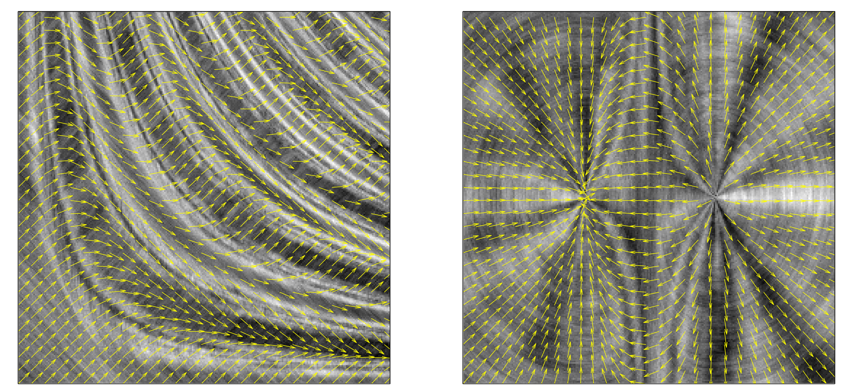


Figure 5: Representation of the 18 neighbors $\mathcal{V}(s_0)$ of s_0

LAFBF with Hurst index constant





We now define our new Gaussian model as a local version of the elementary field that we call *Locally Anisotropic Fractional Brownian Field* (LAFBF):

$$B^H{}_{\alpha_0,\alpha}(x) = \int_{\mathbb{R}^2} (e^{ix\cdot\xi} - 1) \frac{\mathbb{1}_{[-\alpha,\alpha]}(\arg\xi - \alpha_0(x))}{\|\xi\|^{H+1}} d\widehat{W}(\xi)$$

α_0 being now a differentiable function on $\mathbb{R}^2$.
The tangent field of a LAFBF (a) at a point x_0 (in red) with prescribed orientation $\alpha_0(x_0)$ (in yellow) is an elementary field (b) with the same orientation.

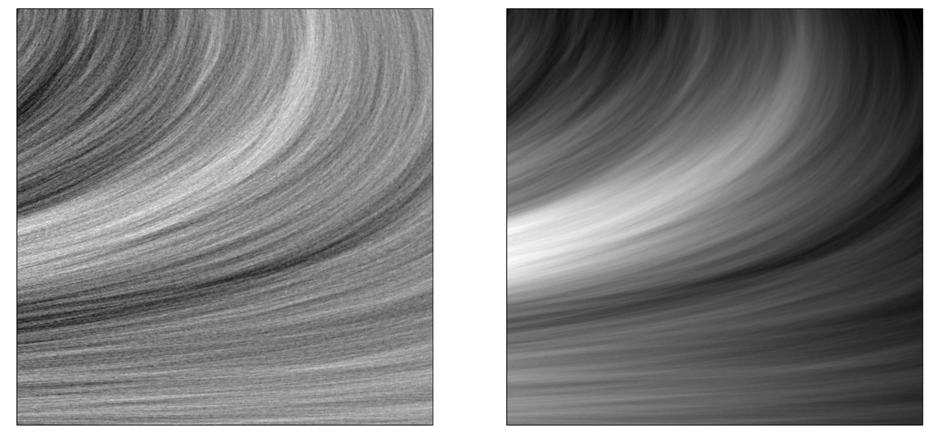


Figure 1: $\alpha_0(x_1, x_2) = -\pi/2 + x_2$, $\alpha = 0.1$, $H \in \{0.1, 0.5\}$

LAFBF with Hurst function $h(x)$

We give a more general definition of LAFBF with the Hurst index varying spatially $x \in \mathbb{R}^2 \mapsto h(x)$

$$B^h{}_{\alpha_0,\alpha}(x) = \int_{\mathbb{R}^2} (e^{ix\cdot\xi} - 1) \frac{\mathbb{1}_{[-\alpha,\alpha]}(\arg\xi - \alpha_0(x))}{\|\xi\|^{h(x)+1}} d\widehat{W}(\xi)$$

Method of simulation

❶ **Generate elementary fields.** Let α_0 be an angle, the synthesis of an elementary field oriented in the direction $\alpha_0^\perp$ can be performed using either the turning band method either the cholesky method by the following covariance with $[\theta_1, \theta_2] \equiv [\alpha_0 - \alpha, \alpha_0 + \alpha]$: $r_{H,\theta_1,\theta_2}(x,y) \equiv \mathrm{Cov}(B^H{}_{\alpha_0,\alpha}(x), B^H{}_{\alpha_0,\alpha}(y)) = v_{H,\theta_1,\theta_2}(x) + v_{H,\theta_1,\theta_2}(y) - v_{H,\theta_1,\theta_2}(x-y)$ where

$$v_{H,\theta_1,\theta_2}(x) = 2^{2H-1}\gamma(H) C_{H,\theta_1,\theta_2}(\arg x)|x|^{2H}$$

$$C_{H,\theta_1,\theta_2}(\theta) = \begin{cases} \beta_H\left(\frac{1-\sin(\theta_2-\theta)}{2}\right) + \beta_H\left(\frac{1-\sin(\theta_1-\theta)}{2}\right), \text{if } \theta_1 \leqslant \theta + \pi/2 \leqslant \theta_2 \\ \beta_H\left(\frac{1+\sin(\theta_2-\theta)}{2}\right) + \beta_H\left(\frac{1+\sin(\theta_1-\theta)}{2}\right), \text{if } \theta_1 \leqslant \theta - \pi/2 \leqslant \theta_2 \\ \left|\beta_H\left(\frac{1-\sin(\theta_2-\theta)}{2}\right) - \beta_H\left(\frac{1-\sin(\theta_1-\theta)}{2}\right)\right|, \text{otherwise} \end{cases}$$

$\beta_H = \int_0^t u^{H-1/2}(1-u)^{H-1/2}$ is the incomplete Beta function and $\gamma(H) = \frac{\pi}{H\Gamma(2H)\sin(H\pi)}$

❷ **Generate N LAFBF oriented by $\alpha_0(x)$ with Hurst index H_n as in Fig.1.** Each LAFBF is computed by the tangent field method : $B^{H_n}{}_{\alpha_0,\alpha}(x_0) = Y_{x_0}(x_0)$ where Y_{x_0} is an H_n Hurst elementary field oriented by $\alpha_0(x_0)$

❷ **Interpolate finally the complete LAFBF by a kriging method** using the N images from the discretization (H_n) of $z \mapsto h(z)$. The kriging relies on the covariance of the LAFBF : $\mathrm{Cov}(B^{h(z_i)}{}_{\alpha_0(x_i,y_i),\alpha}(x_i,y_i), B^{h(z_j)}{}_{\alpha_0(x_j,y_j),\alpha}(x_j,y_j)) = r_{H,\theta_1,\theta_2}((x_i,y_i),(x_j,y_j))$ with parameters :

$$\begin{cases} H = \frac{h(z_i)+h(z_j)}{2} \\ \theta_1 = \max(\alpha_0(x_i,y_i) - \alpha, \alpha_0(x_j,y_j) - \alpha) \\ \theta_2 = \min(\alpha_0(x_i,y_i) + \alpha, \alpha_0(x_j,y_j) + \alpha) \end{cases}$$

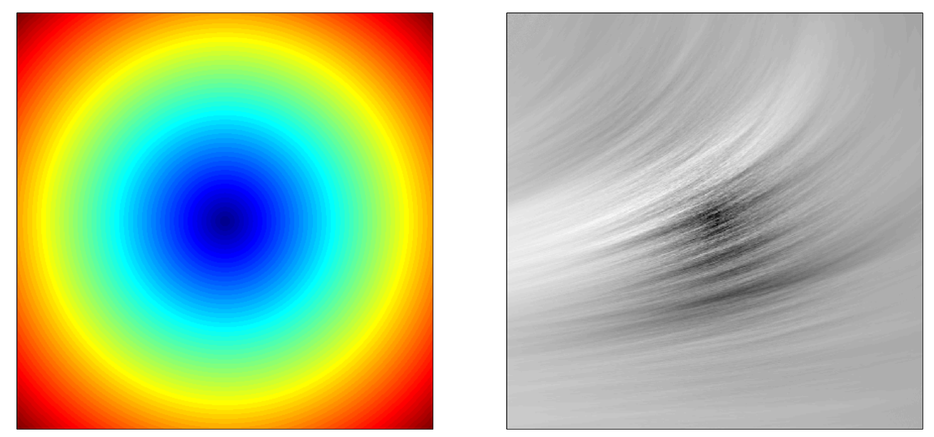


Figure 2: A LAFBF with a radial Hurst function $h(x_1,x_2) = 0.1 + \sqrt{(x_1-0.5)^2 + (x_2-0.5)^2}$