Wavelets and Applications

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M2 MSIAM & Ensimag 3A MMIS

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Under the Wave off Kanagawa

Figure: Hokusai’s most celebrated print called "The Great Wave" (1831)

Credits: British Museum, London. Photograph: Guy Bell/Rex/Shutterstock
Under the Wave off Kanagawa
What are waves about?
What are waves about?
What are waves about?

Love, freedom, health had given
Their ripeness to the manhood of its prime,
And all its pulses beat
Symphonious to the planetary spheres;
Then dulcet music swelled
Concordant with the life-strings of the soul;
It throbbed in sweet and languid beatings there,
Catching new life from transitory death;

Like the vague sighings of a wind at even
That wakes the wavelets of the slumbering sea
And dies on the creation of its breath,
And sinks and rises, falls and swells by fits,
Was the pure stream of feeling
That sprung from these sweet notes,
And o’er the Spirit’s human sympathies
With mild and gentle motion calmly flowed.

Credits: Queen Mab – Percy Bysshe Shelley (1813)
What are waves about?

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Credits: Queen Mab – Percy Bysshe Shelley (1813)
What is a wavelet?

Examples of waves

- Electromagnetic wave
- Radio wave
- Microwave
- Sound wave

Wavelet = "short wave"
From continuous to discrete waves

Credits: The great wave made in LEGO © by Jumpei Mitsui in December 2020.
Natural phenomena may be fractal on several scales
Benefits

At the end of this course, you will be able to:

▶ Understand Fourier limitations
▶ Decompose signals and images onto wavelet bases or frames
▶ Perform multiresolution analysis in order to compress, denoise and analyze signals and images
▶ Have an overview on the potential of wavelets applied to more recent research areas like convolution neural networks (CNN) or graphs analysis
Assessments and guidelines

1. 2 lab sessions using the PyWavelet library

2. 1 project consisting on:
   - Choosing a research article of your choice dealing with wavelets.
   - Reading and understanding the article, writing a summary of what you expect to implement.
   - Practical work: implementation of the method presented in the article.
   - Writing a report including figures of results (in \textit{\LaTeX}).
Assessments and guidelines

- **Ensimag students (not in MSIAM) must form pairs** for lab sessions and the project (precise both your names on the summary)
- Be careful not to choose an article that is neither too simple (close to the lab sessions) nor too difficult (in order to be able to reproduce the results of the paper).
- Since the choice is free, take the opportunity to identify a subject that appeals to you. To get an idea of the wide spectrum of themes invoking the use of wavelets, you’ll find on Chamilo a list of articles chosen by students in previous years.
- Once found, please **mention it in comments on Chamilo’s forum** (« Wavelet project ») which article you’re going to work on (first-come first-served).
- Hand in your assignments on TEIDE
## Deadlines

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Mark scheme

1. **The lab sessions** are each one graded out of **2.5 points**

2. **The project** is graded out of **15 points**
   - Choice of the article (difficulty, length, ...): **1 point**
   - Summary and outline (in line with the targets announced): **2 points**
   - Report redaction (including statement of the method, novelty of the paper, ...): **3 points**
   - Codes (from scratch or existing libraries): **4 points**
   - Numerical results (replication results or extended): **3 points**
   - Interpretations of the results: **2 points**

Penalties:
- **Plagiarism** (copy codes from web or students): mark divided by 2
- **Hand in overdue**: -10% per day
Course materials

Books

Course materials

Links

- **WaveLab** (free Matlab toolbox)
  
  http://www-stat.stanford.edu/~wavelab/

- **A numerical tour of Signal/Image Processing** (by Gabriel Peyré)
  
  http://www.numerical-tours.com/

- **PyWavelets** (python)
  
  https://pywavelets.readthedocs.io/
Course Wavelets and Applications – Outline

1. From Fourier to the 1D Continuous Wavelet Transform
2. Wavelet zoom: a local characterization of functions
3. Lab 1: 1D Continuous Wavelet Transform
4. The 2D Continuous Wavelet Transform
5. The 1D Discrete Wavelet Transform and Multi-resolution Analysis
6. The 2D Discrete Wavelet Transform and Multi-resolution Analysis
7. Approximation in wavelet bases (sparsity, compression, denoising)
8. Lab 2: Fast Wavelet Transform, image compression and denoising
9. The Laplacian of a graph and its applications
10. The graph Fourier transform and wavelets on graphs
11. Lab 3: Applications of Fourier and wavelets on graphs
12. Lab 4: Dedicated to the project
From Fourier to the 1D Continuous Wavelet Transform
A success story

• **Wavelets** for Data representation

• **Wavelets** for numerical simulation

Divergence-free wavelet  Direct Simulation of Turbulence

Credits: Valérie Perrier
A success story

WSQ (1993) is the FBI's Wavelet Scalar Quantization: it is a national standard for the collecting, encoding, storing, and retrieving digitized fingerprint images.

JPEG 2000 is an image coding system that uses state-of-the-art compression techniques based on wavelet technology.

Academy Sci-Tech Award 2013
Awarded to Theodore Kim, Nils Thuerey, Dr. Markus Gross and Doug James for the invention, publication and dissemination of “Wavelet Turbulence” software.

Credits: Valérie Perrier
A success story

- **Abel Prize (2017)**: Yves Meyer, for his pivotal role in the development of the mathematical theory of wavelets.

"Wavelet analysis has been applied in a wide variety of arenas as diverse as applied and computational harmonic analysis, data compression, noise reduction, medical imaging, archiving, digital cinema, deconvolution of the Hubble space telescope images, and the recent LIGO detection of gravitational waves created by the collision of two black holes."

[http://www.abelprize.no/]

The Abel Lecture (Yves Meyer)

www.youtube.com/watch?v=wxmzHwd3z34

Credits: Valérie Perrier
From the music of the spheres to the chirp of black holes

A small detour through old cosmology to meet Fourier

Back to the time of Ptolemy...
Who Wants to Be a Millionaire?
A pre-Copernican TV show

Credits: "Qui veut gagner des millions" (http://www.youtube.com/watch?v=ekmtqODjrSI)
Why was Ptolemy’s system so efficient?

Credits: Joshua Hershey & Universe Today Today
Because of that...

Credits: Carman & Serra (http://www.youtube.com/watch?v=QVuU2YCwHjw)
... or more exactly thanks to him: Joseph Fourier

Just like Mr Jourdain speaking prose, astronomers made Fourier series without realizing it

Credits: 3blue1brown (http://www.youtube.com/watch?v=-qgreAUpPwM)
Fourier series: an intuition behind the decomposition

Periodic signals can be decomposed onto the Fourier basis

\[ f(t) = \cdots + c_{-4}e^{-2 \cdot 2 \pi it} + c_{-1}e^{-1 \cdot 2 \pi it} + c_{0}e^{0 \cdot 2 \pi it} + c_{1}e^{1 \cdot 2 \pi it} + c_{2}e^{2 \cdot 2 \pi it} + \cdots \]

Credits: 3blue1brown (http://www.youtube.com/watch?v=r6sGWTCMz2k)
"With four parameters I can fit an elephant, and with five I can make him wiggle his trunk."

John von Neumann

Credits: El Jj (http://www.youtube.com/watch?v=uazPP0ny3XQ)
Fourier series: is it only useful for drawing?

How Joseph Fourier solved the heat equation

1) Sine = Nice

2) Linearity

3) Fourier series

Credits: 3blue1brown (http://www.youtube.com/watch?v=r6sGWTCMz2k)
Fourier series analysis

The **Fourier analysis** decomposes a signal (function) \( f(x) \) \((x = \text{times})\) into a sum of sinusoidal functions:

- For a \( T \)-periodic function \( f \), with \( f \in L^2(0, T) \):
  
  \[
  f(x) = \sum_{n \in \mathbb{Z}} c_n(f) e^{2i\pi \frac{n}{T} x} \quad \text{(synthesis)}
  \]

  where the Fourier coefficients are:

  \[
  c_n(f) = \frac{1}{T} \int_{0}^{T} f(x) e^{-2i\pi \frac{n}{T} x} \, dx \quad \text{(analysis)}
  \]

  are related to the frequency \( \frac{n}{T} \) (in Hz).

**Parseval equality:**

\[
\sum_{n \in \mathbb{Z}} |c_n(f)|^2 = \frac{1}{T} \int_{0}^{T} |f(x)|^2 \, dx \quad \text{(energy conservation)}
\]
Fourier series limitations
Discontinuities require a lot of sinusoids to be described

\[ f(x) = \begin{cases} 
-1 & \text{if } -\pi \leq x < 0 \\
+1 & \text{if } 0 \leq x < \pi 
\end{cases} = \sum_{n=1}^{+\infty} \frac{4}{\pi(2n-1)} \sin((2n-1)x) \]

Fourier transform

- For a function $f \in L^2(\mathbb{R})$:

  $$f(x) = \int_{-\infty}^{+\infty} \hat{f}(\nu)e^{2i\pi \nu x} \, d\nu$$  
  (synthesis)

  where the Fourier transform of $f$ is:

  $$\hat{f}(\nu) = \int_{-\infty}^{+\infty} f(x)e^{-2i\pi \nu x} \, dx$$  
  (analysis)

  gives information on $f$ for the frequency $\nu$.

Plancherel-Parseval equality:

$$\int_{-\infty}^{+\infty} |\hat{f}(\nu)|^2 \, d\nu = \int_{-\infty}^{+\infty} |f(x)|^2 \, dx$$  
  (energy conservation)
Fourier transform: an intuition behind the transformation
aperiodic signals can also be decomposed onto the continuous dictionary of exponentials

\[ \hat{g}(f) = \int_{t_1}^{t_2} g(t) e^{-2\pi if t} dt \]

Credits: 3blue1brown (https://www.youtube.com/watch?v=spUNpyF58BY)
Fourier Transform visualization

Wrap the signal around a circle

\[ \hat{g}(f) = \frac{1}{N} \sum_{k=1}^{N} g(t_k) e^{-2\pi i ft_k} \]

To find the energy at a particular frequency, the signal is wrapped around a circle at the particular frequency and the points along the path are averaged.

Credits: Elan Ness-Cohn
Fourier transform limitations

Example: two musical notes played at the same time

Figure: Signal $f(x) = \sin(40\pi x) + \sin(170\pi x)$ (top), and modulus of its Fourier Transform $\hat{f}(\nu)$ (bottom)
Fourier transform limitations

Example of two musical notes played one after the other

The frequency analysis do not inform on the transient phenomenon in the signal ⇒ Loss of temporal localization
Fourier cat transformation

Hi, Dr. Elizabeth? Yeah, uh... I accidentally took the Fourier transform of my cat...

Meow!
Take home message

To sum up

- Periodic functions (as planets motion along closed orbit) can be approximated by epicycles, that is by Fourier series.
- Fitting data do not necessarily mean that the mechanics behind is understood, and saving the phenomena can lead to a kind of overfitting. What is a good model or a good theory?
- The relativity of motion makes possible to consider different coordinate systems to describe trajectories. Something which is well known by physicists: the choice of the frame of reference can greatly simplify mathematical calculations.
- An appropriate representation of the signal can also reduced the number of parameters needed to encode its information.

⇒ Toward a sparse representation of signals
Take home message

To sum up

- Fourier series decomposition allowed Joseph Fourier to solve partial differential equations (heat equation).
- Extension to the Fourier transform for aperiodic signal also reveals the frequency contents of the signal, but suffer of the same issues:
  - Discontinuities involve a lot of significant coefficients in the decomposition, whose the decrease in amplitudes encodes the global regularity of the signal.
  - Losing the temporal localization, the Fourier transform does not allow to capture transient phenomena in the signal.

$\Rightarrow$ Toward a time-frequency representation of signals
Short Time Fourier Transform (STFT)

Multiplication of the signal $f(x)$ by a window $w(x - b)$ (real and of size $a_0$) and computation of the Fourier transform of this product:

$$Sf(\nu, b) = \int_{-\infty}^{+\infty} f(x)w(x - b) e^{-2i\pi\nu x} \, dx$$

where $b$ represents time and $\nu$ frequency. $f$ can be recovered from its STFT coefficients:

$$f(x) = C_h \int\int_{\mathbb{R}^2} Sf(\nu, b)w(x - b)e^{2i\pi\nu x} \, d\nu \, db$$
Special case: the Gabor Transform

- In the Short Time Fourier Transform
  \[
  S_f(\nu, b) = \int_{-\infty}^{+\infty} f(x) w(x - b) e^{-2i\pi\nu x} \, dx = \langle f, \psi_{\nu,b} \rangle
  \]
  the analyzing functions are:

  \[
  \psi_{\nu,b} = w(x - b) e^{2i\pi\nu x}
  \]

- In the **Gabor transform** (1946) the window \(w\) is a Gaussian of scale \(\sigma\):
  \[
  w(x) = \frac{1}{\sigma} e^{-\pi \left(\frac{x}{\sigma}\right)^2}
  \]
  and the Gabor functions are then (\(\sigma = 1\)):

  \[
  g_{\nu,b} = e^{-\pi (x-b)^2} e^{2i\pi\nu x}
  \]

(a) \(\nu = 2\)  
(b) \(\nu = 5\)  
(c) \(\nu = 15\)
The time-frequency analysis allows to recover both frequencies (the notes) and temporal information (the temporal order) of the signal $f_1$:

Figure: Time-frequency plane with $b$ on the x-axis and $\nu$ on the y-axis, representing the density energy $|Sf(\nu, b)|^2 = |\langle f, g_{\nu, b} \rangle|^2$ called the spectrogram.
Short Time Fourier Transform

Analogy with music scores: an example with a piano

Credits: Patrick Flandrin, "Au-delà de Fourier, un monde qui vibre" (interstices.info)
Heisenberg boxes
Time-frequency localization and spread

\[ g_{\xi, b}(t) = w(t - b)e^{i\xi t} \longleftrightarrow \hat{g}_{\xi, b}(\omega) = \hat{w}(\omega - \xi)e^{-ib(\omega - \xi)} \]

\[ \sigma_t^2 = \int_{-\infty}^{\infty} (t - b)^2 |g_{\xi, b}(t)|^2 dt = \int_{-\infty}^{\infty} t^2 |w(t)|^2 dt \]

\[ \sigma_{\omega}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\omega - \xi)^2 |\hat{g}_{\xi, b}(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |\hat{w}(t)|^2 d\omega \]
Heisenberg boxes

Example: Gabor limits

\[ g_{s,f}(t) = w(t - s)e^{j2\pi ft} \]

\[ w(t) = (\pi \sigma^2)^{-1/4}e^{-t^2/2\sigma^2} \]

Credits: Pierre Chainais, "De la transformée de Fourier à l’analyse temps-fréquence bivariée"
Heisenberg boxes
Time-frequency localization and spread

- Can we construct a function $f$, with an energy that is highly localized in time and with a Fourier transform $\hat{f}$ having an energy concentrated in a small-frequency interval?

- To reduce the time spread of $f$, we can scale it by $a < 1$, while keeping its total energy constant:

$$f_a(t) = \frac{1}{\sqrt{a}} f \left( \frac{t}{a} \right), \quad \| f_a \|^2 = \| f \|^2$$

- The corresponding Fourier transform is dilated by a factor $1/a$:

$$\hat{f}_a(\omega) = \sqrt{a} \hat{f}(a\omega)$$

$\Rightarrow$ So we lose in frequency localization what we gained in time. Underlying is a trade-off between time and frequency localization.

Credits: S. Mallat (Wavelet tour)
Heisenberg’s indeterminacy relations

Defining the average location and frequency respectively by:

$$b = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} t|f(t)|^2dt, \quad \xi = \frac{1}{2\pi\|f\|^2} \int_{-\infty}^{\infty} \omega|\hat{f}(\omega)|^2d\omega$$

The variances around these average values are respectively:

$$\sigma_t^2 = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} (t-b)^2|f(t)|^2dt, \quad \sigma_\omega^2 = \frac{1}{2\pi\|f\|^2} \int_{-\infty}^{\infty} (\omega-\xi)^2|\hat{f}(\omega)|^2d\omega$$

Theorem (Heisenberg’s indeterminacy relations)

The temporal variance and the frequency variance of \( f \in L^2(\mathbb{R}) \) satisfy

$$\sigma_t \sigma_\omega \geq \frac{1}{2}$$

This inequality is an equality iff \( \exists (b, \xi, c_1, c_2) \in \mathbb{R}^2 \times \mathbb{C}^2 \) such that

$$f(t) = c_1e^{i\xi t - c_2(t-b)^2}$$

Credits: S. Mallat
Heisenberg’s indeterminacy relations

**Proof** *(Weyl)*: this proof supposes that \( \lim_{|t| \to +\infty} \sqrt{t} f(t) = 0 \) \((*)\) but the theorem is valid for any \( f \in L^2(\mathbb{R}) \). The average time and frequency location of \( e^{-i\xi t} f(t + b) \) is zero. Thus, it is sufficient to prove the theorem for \( b = \xi = 0 \).

Since \( f'(t)(\omega) = i\omega \hat{f}(\omega) \), the Plancherel identity applied to \( i\omega \hat{f}(\omega) \) yields

\[
\sigma_t^2 \sigma_\omega^2 = \frac{1}{\|f\|^4} \left( \int_{-\infty}^{\infty} |t f(t)|^2 dt \right) \left( \int_{-\infty}^{\infty} |f'(t)|^2 dt \right) \quad (**)
\]

Schwarz’s inequality and the assumption \((*)\) [for the last equality] imply

\[
\sigma_t^2 \sigma_\omega^2 \geq \frac{1}{\|f\|^4} \left( \int_{-\infty}^{\infty} |t f'(t) f^*(t)| dt \right)^2 \quad \forall z \in \mathbb{C}, \ |z| \geq \Re(z) = \frac{z + z^*}{2}
\]

\[
\geq \frac{1}{\|f\|^4} \left( \int_{-\infty}^{\infty} \frac{t}{2} (f'(t) f^*(t) + f'^*(t) f(t)) dt \right)^2
\]

\[
\geq \frac{1}{4\|f\|^4} \left( \int_{-\infty}^{\infty} t (|f(t)|^2)' dt \right)^2 \quad \text{IBPF} \quad \frac{1}{4\|f\|^4} \left( \int_{-\infty}^{\infty} |f(t)|^2 dt \right)^2 = \frac{1}{4}
\]

Credits: S. Mallat (Wavelet tour) \( \text{(IBPF = Integration By Parts Formula)} \)
Heisenberg’s indeterminacy relations

**Proof**: To obtain an equality, Schwarz’s inesquality applied to (**) must be an equality. This implies that there exists \( c_2 \in \mathbb{C} \) such that

\[
f'(t) = -2c_1 tf(t)
\]

Thus, there exists \( c_1 \in \mathbb{C} \) such that

\[
f(t) = c_1 e^{-c_2 t^2}
\]

When \( b \neq 0 \) and \( \xi \neq 0 \) a time and frequency translation yield the result.

**Remark**: motivated by quantum mechanics, Gabor proposed time-frequency atoms that have a minimal spread in a time-frequency plane. By showing that signal decompositions over the dictionary of Gabor atoms are closely related to our perception of sounds, and that they exhibit important structures in speech and music recordings, he demonstrated the importance of localized time-frequency signal processing.

Credits: S. Mallat (Wavelet tour)
Heisenberg’s indeterminacy relations
Some intuitions behind

\[
\text{TF}(f \cdot \Pi_{[-a/2, a/2]}) = \text{TF}(f) \ast \text{TF}(\Pi_{[-a/2, a/2]}) = \text{TF}(f) \ast a\text{sinc}(\pi a \cdot)
\]

Credits: 3blue1brown (http://www.youtube.com/watch?v=MBnnXbOM5S4)
Heisenberg’s indeterminacy relations
Some intuitions behind

\[ \Delta f \] decreases as the measurement interval \( \Delta t \) increases, and vice versa.

**Figure:** Improved frequency measurement over longer time intervals. The uncertainty in the frequency \( \Delta f \) decreases as the measurement interval \( \Delta t \) increases, and vice versa.

Credits: Bruce MacLennan (Gabor Representation)
Heisenberg’s indeterminacy relations
Some intuitions behind

Figure: Measuring frequency by counting maxima in a given time interval. The circled numbers indicate the maxima counted during the measurement interval $\Delta t$. Since signals of other frequencies could also have the same number of maxima in that interval, there is an uncertainty $\Delta f$ in the frequency.

Credits: Bruce MacLennan (Gabor Representation)
Heisenberg’s indeterminacy relations
Some intuitions behind

**Figure:** Minimum time interval $\Delta t$ to detect frequency difference $\Delta f$. If two signals differ in frequency by $\Delta f$, then a measurement of duration $\Delta t \geq 1/\Delta f$ is required to guarantee a difference in counts of maxima. (Italic numbers indicate maxima of signal of frequency $f$, roman numbers indicate maxima of signal of higher frequency $f + \Delta f$)

\[(f + \Delta f)\Delta t - f\Delta t \geq 1 \iff \Delta f \Delta t \geq 1\]

Credits: Bruce MacLennan (Gabor Representation)
Short Time Fourier Transform

Examples

1. A sinusoidal wave \( f(t) = e^{i\xi_0 t} \) whose Fourier transform is a Dirac \( \hat{f}(\omega) = 2\pi \delta(\omega - \xi_0) \) has a STFT:

\[
S_f(\xi, b) = \hat{w}(\xi - \xi_0)e^{-ib(\xi - \xi_0)}
\]

Its energy is spread over the frequency interval

\[
\xi \in [\xi_0 - \sigma_\omega/2, \xi_0 + \sigma_\omega/2]
\]

2. A Dirac \( f(t) = \delta(t - b_0) \) has a STFT:

\[
S_f(\xi, b) = w(b - b_0)e^{-i\xi b_0}
\]

Its energy is spread in the time interval

\[
b \in [b_0 - \sigma_t/2, b_0 + \sigma_t/2]
\]
Limitation of the Short Time Fourier Transform

The STFT cannot separate events of a distance smaller than $a_0$, that is to localize the two frequencies and the transient phenomena.

Figure: Signal $f_2 = f_1 + \delta_1 + \delta_2$ and its Gabor transform with $a_0 = 0.05$
Limitation of the Short Time Fourier Transform

The STFT cannot separate events of a distance smaller than $a_0$, that is to localize the two frequencies and the transient phenomena.

Figure: Signal $f_2 = f_1 + \delta_1 + \delta_2$ and its Gabor transform with $a_0 = 0.005$
Pioneer works on wavelets

- **Jean Morlet** research engineer at ELF Aquitaine discovered wavelets for solving signal processing problems arising from oil exploration.

- **Alex Grossmann** recognized in the Morlet wavelets something similar to coherent states formalism in quantum mechanics and developed an exact inversion formula for the wavelet transform.

- They developed the mathematics of the continuous wavelet transforms in their article: "Decomposition of Hardy Functions into Square Integrable Wavelets of Constant Shape" (1984)
"Gaborettes" vs Morlet wavelets

Figure: (Left) Gabor $\psi_{a,b}(t) = e^{it/a}\psi(t-b)$, (right) Morlet $\psi_{a,b}(t) = a^{-1/2}\psi\left(\frac{t-b}{a}\right)$

**Gabor** ⇒ frequency modulation inside a constant window width

**Wavelets** ⇒ shape of $\psi_{a,b}$ doesn’t change, simply dilated or compressed
The Continuous Wavelet Transform (CWT) – Definition

\[
Wf(a, b) = \int_{-\infty}^{+\infty} f(x) \psi_{a,b}(x) \, dx = \langle f, \psi_{a,b} \rangle, \quad a > 0, \quad b \in \mathbb{R}
\]

The analyzing functions or **wavelets** are defined by:

\[
\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \psi \left( \frac{x - b}{a} \right)
\]
Wavelet family in physical space
Example: the Morlet wavelets

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x - b}{a}\right)$$

with mother wavelet

$$\psi(x) = \cos(x)e^{-10\pi x^2}$$

Figure: Morlet wavelets of scale: $a = 1/2, 1, 2$ (real part). The scale $a$ gives the support size (inverse of a frequency), whereas $b$ gives the position.
Wavelet analysis of the toy signal with Morlet wavelets

Figure: Signal $f_2$ (two notes + scratch) and its CWT
Wavelet definition

A function \( \psi(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) is a **wavelet** if it satisfies the following **admissibility condition**:

\[
C_\psi = \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\nu)|^2}{|\nu|} \, d\nu < \infty
\]

which implies \( \int_{-\infty}^{+\infty} \psi(x) \, dx = 0 \) (and this is equivalent if \( x\psi \) integrable).

**Examples**

1. **The (complex) Morlet wavelet**
   - Mother wavelet: \( \psi(x) = e^{-\pi x^2} e^{10i\pi x} \)
   - Its Fourier Transform: \( \hat{\psi}(\nu) = e^{-\pi(\nu-5)^2} \)

2. **Gaussian derivatives**
   - Mother wavelet: \( \psi_n(x) = \frac{d^n}{dx^n} e^{-\pi x^2}, \quad n \geq 1 \)  
     (for \( n = 2, \psi_2 \) is called the "Mexican Hat")
   - Its Fourier Transform: \( \hat{\psi}_n(\nu) = (2i\pi \nu)^n e^{-\pi \nu^2} \)
Wavelet analysis: a picture is worth a thousand words

Figure: Correlations of the signal $f$ with $\psi_{s,u}$ the Morlet wavelet dilated by $s$ and translated by $u$ (curves in blue on the top) whose coordinates $(u, \log_2(s))$ (represented by the blue points on the bottom) of the scalogram contain the value $\langle f, \psi_{s,u} \rangle$

Credits: S. Mallat
Fourier Transform of wavelets

\[ \psi_{a,b}(x) = \frac{1}{\sqrt{a}} \psi \left( \frac{x - b}{a} \right) \quad \leftrightarrow \quad \hat{\psi}_{a,b}(\nu) = \sqrt{a} \hat{\psi}(a\nu)e^{-2i\pi b\nu} \]

**Figure:** Fourier Transform (modulus) of Morlet wavelets of scales \( a = 1/2, 1, 2 \). Wavelets behaves as band-pass filters around frequency \( \nu = \frac{\nu_0}{a} \), where \( \nu_0 \) is the peak wavenumber (max of \( \hat{\psi} \)). For the Morlet wavelet, \( \nu_0 = 5 \).
Equivalent definition

Let \( f \in L^2(\mathbb{R}) \). For all \( a > 0, \ b \in \mathbb{R} \),

\[
Wf(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x) \psi \left( \frac{x - b}{a} \right) \, dx
\]

\[
Wf(a, b) = \sqrt{a} \int_{-\infty}^{+\infty} \hat{f}(\nu) \hat{\psi}(a\nu) e^{2\pi i \nu b} \, d\nu
\]

Proof: From the Parseval formula

\[
Wf(a, b) = \langle f, \psi_{a,b} \rangle = \langle \hat{f}, \hat{\psi}_{a,b} \rangle
\]

- In the time domain (\( x \)), \( Wf(a, b) \) provides information on the signal \( f \) around point \( b \) in a vicinity of size \( \sim a \).
- In the frequency domain (\( \nu \)), \( Wf(a, b) \) provides information on the signal \( \hat{f} \) around frequency \( \sim \frac{1}{a} \).

\( \Rightarrow \) Wavelet analysis is a time-scale analysis
Time-frequency resolution of wavelets

\[
\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x - b}{a}\right) \quad \leftrightarrow \quad \hat{\psi}_{a,b}(\nu) = \sqrt{a} \hat{\psi}(a\nu)e^{-2i\pi b\nu}
\]

We suppose that \( \psi \) is analytic, \( \psi(0) = 0 \) and \( \eta = \frac{1}{2\pi} \int_0^\infty \omega |\hat{\psi}(\omega)|^2 d\omega \)

\[
\int_{-\infty}^{\infty} (t - b)^2 |\psi_{a,b}(t)|^2 dt \overset{t \leftarrow \frac{t-b}{a}}{=} a^2 \int_{-\infty}^{\infty} a^2 t^2 |\psi(t)|^2 dt = a^2 \sigma_t^2
\]

\[
\frac{1}{2\pi} \int_0^\infty \left(\omega - \frac{\eta}{a}\right)^2 |\hat{\psi}_{\xi,b}(\omega)|^2 d\omega = \frac{1}{2\pi a^2} \int_0^\infty (\omega - \eta)^2 |\hat{\psi}(\omega)|^2 d\omega = \frac{\sigma_\omega^2}{a^2}
\]

The energy spread of a wavelet time-frequency atom \( \psi_{a,b} \) corresponds to a Heisenberg box centered at \( (b, \xi = \eta/a) \), of size \( a \sigma_t \) along time and \( \sigma_\omega/a \) along frequency. The area of the rectangle remains equal to \( \sigma_t \sigma_\omega \) at all scales but the resolution in time and frequency depends on \( a \). An analytic wavelet transform defines a local time-frequency energy density

\[
P_{Wf}(b, \xi) = |Wf(a, b)|^2 = \left|Wf\left(\frac{\eta}{\xi}, b\right)\right|^2 \quad \text{(scalogram)}
\]
Heisenberg boxes of two wavelets $\psi_{a,b}$ and $\psi_{a_0,b_0}$

$$\int_{-\infty}^{\infty} (t - b)^2 |\psi_{a,b}(t)|^2 \, dt = a^2 \sigma_t^2$$

$$\frac{1}{2\pi} \int_{0}^{\infty} \left( \omega - \frac{\eta}{a} \right)^2 |\hat{\psi}_{\xi,b}(\omega)|^2 \, d\omega = \frac{\sigma_\omega^2}{a^2}$$
Inversion of the Continuous Wavelet Transform

Synthesis formula and energy conservation

**Theorem (Calderón, Grossmann and Morlet)**

Let $\psi \in L^2(\mathbb{R})$ be a real function such that

$$C_\psi = \int_{-\infty}^{+\infty} \frac{\left| \hat{\psi}(\nu) \right|^2}{|\nu|} \, d\nu < \infty$$

Any $f \in L^2(\mathbb{R})$ satisfies

$$f(x) = \frac{1}{C_\psi} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} Wf(a, b) \psi_{a,b}(x) \frac{da \, db}{a^2} \tag{*}$$

and

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx = \frac{1}{C_\psi} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} |Wf(a, b)|^2 \frac{da \, db}{a^2} \tag{**}$$
Proof \textit{(Synthesis formula)}: For a fixed $a$, the CWT can be written:

\[
Wf(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x) \psi\left(\frac{x - b}{a}\right) dx = (f \ast \tilde{\psi}_a)(b)
\]

where we have noted:

\[
\psi_a(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x}{a}\right), \quad \tilde{\psi}_a(x) = \psi_a(-x)
\]

The right integral $b(x)$ of $(\ast)$ can now be rewritten as a sum of convolutions:

\[
b(x) = \frac{1}{C_\psi} \int_0^{+\infty} Wf(a, \cdot) \ast \psi_a(x) \frac{da}{a^2} = \frac{1}{C_\psi} \int_0^{+\infty} f \ast \tilde{\psi}_a \ast \psi_a(x) \frac{da}{a^2}
\]

\[
\hat{b}(\omega) = \frac{1}{C_\psi} \int_0^{+\infty} \hat{f}(\omega) \sqrt{a} \hat{\psi}^*(a \omega) \sqrt{a} \hat{\psi}(a \omega) \frac{da}{a^2} = \frac{\hat{f}(\omega)}{C_\psi} \int_0^{+\infty} |\hat{\psi}(a \omega)|^2 \frac{da}{a}
\]

By the change of variable $\xi = a \omega$ we get $\hat{b}(\omega) = \frac{\hat{f}(\omega)}{C_\psi} \int_0^{+\infty} |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} = \hat{f}(\omega)$

The equality of their Fourier transform leads to $b = f$. QED \hfill \blacksquare
Inversion with a different synthesis wavelet

- **Decomposition** with an *analysing wavelet* \( g \): \( a > 0, \ b \in \mathbb{R} \),

  \[
  W_g f(a, b) = \int_{-\infty}^{+\infty} f(x) \frac{1}{\sqrt{a}} \tilde{g} \left( \frac{x - b}{a} \right) \, dx
  \]

- **Synthesis** with a *reconstruction wavelet* \( h \):

  \[
  f(x) = \frac{2}{c_{gh}} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} W_g f(a, b) \frac{1}{\sqrt{a}} h \left( \frac{x - b}{a} \right) \frac{da}{a^2} \, db
  \]

- **Cross-admissibility condition** on wavelets \( g \) et \( h \) (\( g, h \in L^2(\mathbb{R}) \)):

  \[
  c_{gh} = \int_{-\infty}^{+\infty} \frac{\hat{g}(k)\hat{h}(k)}{|k|} \, dk < +\infty
  \]

**Remark:** In this case, only \( h \) or \( g \) has to be a zero mean function.
For a fixed $a$, the CWT is a convolution product:

$$W_f(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x) \psi\left(\frac{x - b}{a}\right) \, dx$$

where we have noted:

$$\tilde{\psi}_a(x) = \frac{1}{\sqrt{a}} \psi\left(-\frac{x}{a}\right)$$
Coding – In practice

\[ f \ast \hat{\psi}_a : b \mapsto \mathcal{W}f(a, b) \]

3D plot of Wavelet Transform

2D plot of Wavelet Transform

source: ataspinar.com
Examples and Interpretation

Example 1: pure cosine

If \( f \) is a **pure cosine** \( f(x) = \cos(2\pi kx) \), then

\[
Wf(a, b) = \int_{-\infty}^{+\infty} \left( \frac{e^{2i\pi kx} + e^{-2i\pi kx}}{2} \right) \psi_{a,b}(x) \, dx
\]

\[
= \frac{1}{2} \left[ \hat{\psi}_{a,b}(k) + \hat{\psi}_{a,b}(-k) \right]
\]

\[
= \sqrt{a} \left[ \hat{\psi}(ak)e^{2i\pi kb} + \hat{\psi}(-ak)e^{-2i\pi kb} \right]
\]

- If the wavelet \( \psi \) is analytic complex:

\[
Wf(a, b) = \frac{\sqrt{a}}{2} \hat{\psi}(ak)e^{2i\pi kb}
\]

- If the wavelet \( \psi \) is real \( Wf(a, b) = \sqrt{a} \text{ Re} \left( \hat{\psi}(ak)e^{-2i\pi kb} \right) \)
Example 1: pure cosine

Pure cosine \( f(x) = \cos(20\pi x) \) \((k = 10)\)

Figure: CWT (modulus), using the Morlet wavelet (analytic complex)
Example 1: pure cosine

Pure cosine \( f(x) = \cos(20\pi x) \) (\( k = 10 \))

Figure: CWT (modulus), using the Gaussian derivatives (real)
Examples and Interpretation

Example 2: a Dirac

If $f$ is a **Dirac** $f(x) = \delta(x - x_0)$ (pointwise measure supported by $x_0$), then:

$$Wf(a, b) = \int_{-\infty}^{+\infty} \delta(x - x_0) \frac{\psi_{a,b}(x)}{\psi_{a,b}(x_0)} \, dx$$

$$= \frac{1}{\sqrt{a}} \psi \left( \frac{x_0 - b}{a} \right)$$

**Remark:** At each scale $a, b \to Wf(a, b)$ is the wavelet of scale $a$ centered on $x_0$ (up to a symmetry).
Example 2: a Dirac

Dirac $\delta_{x_0}$

Figure: Signal "Dirac" and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)
Example 3: the periodic square wave

Periodic square wave

Figure: Square wave and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)
Example 4: a modulated wave

Modulated wave

Figure: Modulated wave and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)
Example 5: 2 sinusoids with noise

2 sinusoids with noise

Figure: Signal and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)
Example 6: Holder function of exponentant $\frac{1}{2}$

Holder function of exponentant $\frac{1}{2}$

Figure: $f(x) = \sqrt{|\cos(2\pi x)|}$ and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)
The Dyadic Wavelet Transform

Fig. 6.1. A Wavelet Tour of Signal Processing, 3rd ed. Wavelet transform $W_f(u, s)$ calculated with $\varphi = \mathbf{1}$ where $\varphi$ is a Gaussian, for the signal $f$ shown above. The position parameter $u$ and the scale $s$ vary respectively along the horizontal and vertical axes. Black, grey and white points correspond respectively to positive, zero and negative wavelet coefficients. Singularities create large amplitude coefficients in their cone of influence.
The Dyadic Wavelet Transform

For a fixed $a = 2^j$, the Dyadic Wavelet Transform is a convolution product:

$$Wf(2^j, b) = \frac{1}{\sqrt{2^j}} \int_{-\infty}^{+\infty} f(x) \psi\left(\frac{x - b}{2^j}\right) \, dx$$

$$d_j(b) = (f \ast \tilde{\psi}_j)(b)$$

where we have noted:

$$\psi_j(x) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{x}{2^j}\right), \quad \tilde{\psi}_j(x) = \psi_j(-x)$$
The Dyadic Wavelet Transform

For a fixed $a = 2^j$, the Dyadic Wavelet Transform is a convolution product:

$$Wf(2^j, b) = \frac{1}{\sqrt{2^j}} \int_{-\infty}^{+\infty} f(x) \psi\left(\frac{x - b}{2^j}\right) \, dx$$

$$d_j(b) = (f \ast \hat{\psi}_j)(b)$$

whose Fourier transform is

$$\hat{d}_j(\omega) = \hat{f}(\omega) \sqrt{2^j} \hat{\psi}^*(2^j \omega)$$
The Dyadic Wavelet Transform

Theorem (Littlewood-Paley, 1930)

If \( \sum_j |\hat{\psi}(2^j \omega)|^2 = 1 \) then

\[
f(x) = \sum_j 2^{-j} \int Wf(2^j, b) \psi_{2^j, b}(x) \, db
\]

**Proof**: Remark that

\[
\int Wf(2^j, b) \psi_{2^j, b}(x) \, db = d_j \ast \psi_j(x)
\]

then take the Fourier transform

\[
\sum_j 2^{-j} d_j(\omega) \hat{\psi}_j(\omega) = \sum_j 2^{-j} \hat{f}(\omega) \sqrt{2^j} \hat{\psi}^*(2^j \omega) \sqrt{2^j} \psi(2^j \omega)
\]

\[
= \hat{f}(\omega) \sum_j |\hat{\psi}(2^j \omega)|^2 = \hat{f}(\omega)
\]

\[= 1\]
Take home message

Time-frequency vs time-scale

(a) Diracs

(b) Fourier

(c) STFT

(d) Wavelets dyadics
Take home message
Heisenberg for wavelets

Credits: Paul S Addison’s figure modified
Take home message

Scalogram construction

Credits: Paul S Addison’s figure adapted