Wavelets and Applications

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M2 MSIAM & Ensimag 3A MMIS

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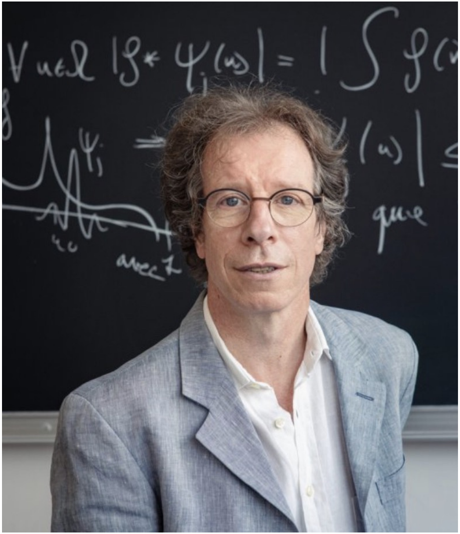
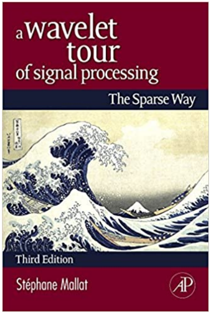
Under the Wave off Kanagawa

Figure: Hokusai's most celebrated print called "The Great Wave" (1831)

Stéphane Mallat is awarded the 2025 CNRS Gold Medal





What are waves about?

“We often imagine mathematics as a body of abstract concepts that apply ‘from above’ to reality. But most of the time, it’s the other way around: problems from the real world force us to build new mathematical tools. And to shape them, one has to ‘get one’s hands dirty,’ building bridges between mathematical abstractions and concrete questions from the world. And it is precisely there, on that boundary between the two, that I feel at home.”

– Stéphane Mallat

What are waves about?

What are waves about?

"With an image, if one wants to be parsimonious, the focus must be on the meaningful variations — an edge, an abrupt change of color, etc. In mathematics, we try to capture the essence of the problem — the pursuit of parsimony — by stepping away from the context of specific applications, in order to find general solutions that can then be applied broadly."

– Stéphane Mallat

What are waves about?

Love, freedom, health had given
Their ripeness to the manhood of its prime,
And all its pulses beat
Symphonious to the planetary spheres;
Then dulcet music swelled
Concordant with the life-strings of the soul;
It throbbed in sweet and languid beatings there,
Catching new life from transitory death;

Like the vague sighings of a wind at even
That wakes the wavelets of the slumbering sea
And dies on the creation of its breath,
And sinks and rises, falls and swells by fits,
Was the pure stream of feeling
That sprung from these sweet notes,
And o'er the Spirit's human sympathies
With mild and gentle motion calmly flowed.

Credits: Queen Mab – Percy Bysshe Shelley (1813)

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What are waves about?

“In writing, with a limited vocabulary, one can certainly express complex ideas, but this often comes at the cost of long-winded circumlocutions and, in the end, approximation. To produce shorter, more impactful sentences, one needs to enrich one’s vocabulary. That is why I introduced the concept of a ‘mathematical dictionary,’ made up of a large number of elementary building blocks, more specialized than wavelets.”

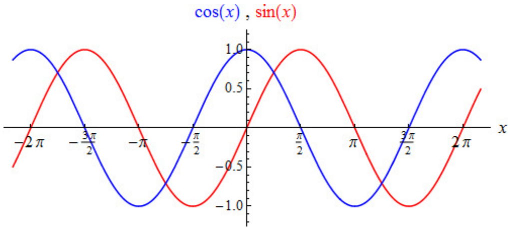
– Stéphane Mallat

What is a wavelet?

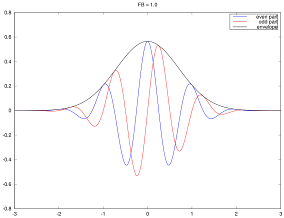
Examples of waves

- Electromagnetic wave
- Radio wave
- Microwave
- Sound wave

Wavelet = "short wave"



sinusoidal waves



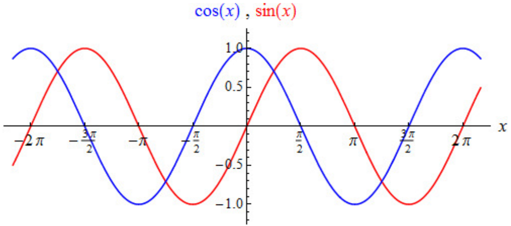
wavelet

Why using wavelets?

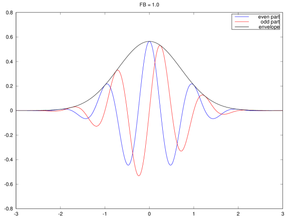
Representing large-scale data as a superposition of a minimal number of elementary structures.

"It's a bit like wanting to build a house out of Lego, using as few bricks as possible while still allowing oneself the freedom to define the shape of those elementary bricks."

– **Stéphane Mallat**



sinusoidal waves

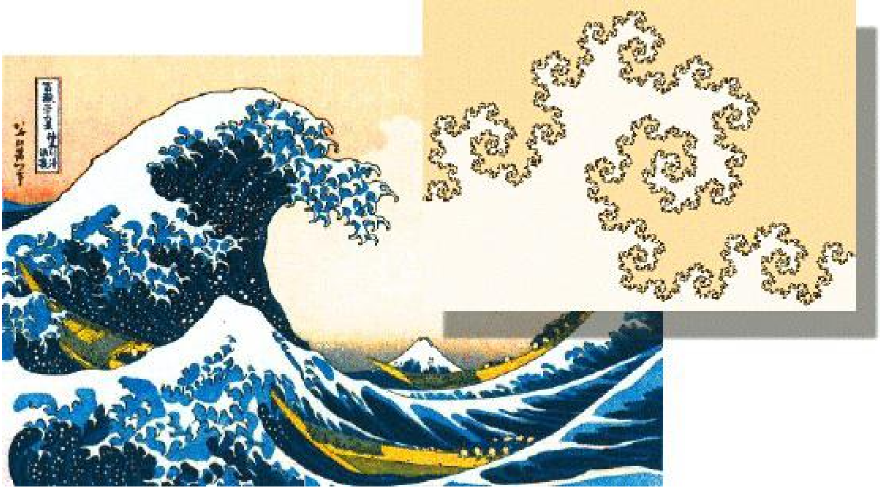


wavelet

From continuous to discrete waves

Credits: The great wave made in LEGO © by Jumpei Mitsui in December 2020.

Natural phenomena may be fractal on several scales



Credits: theorderofnature.wordpress.com

Benefits

At the end of this course, you will be able to:

- Understand the limitations of Fourier analysis
- Decompose signals and images onto wavelet bases or frames
- Perform multiresolution analysis to compress, denoise and analyze signals and images
- Gain an overview of the potential applications of wavelets in emerging research areas like convolution neural networks (CNN) or graphs analysis

Assessments and guidelines

1. Two lab sessions using the PyWavelet library

2. One project consisting on:
   - Selecting a research article of your choice that involves wavelets.
   - Reading and understanding the article, and writing a summary of what you intend to implement.
   - Practical work: implementating the method presented in the article.
   - Writing a report that includes figures and results (using LaTeX).

Assessments and guidelines

- **Students must form pairs** for lab sessions and the project (please include both names on the summary)
- Be careful not to choose an article that is too simple (similar to the lab sessions) or too complex (to ensure you can reproduce the results).
- Since the choice is up to you, take this opportunity to pick a subject that genuinely interests you. To get an idea of the wide range of themes that use wavelets, you'll find a list of articles chosen by students in previous years on Chamilo.
- Once you've found an article, please **mention it in the comments on Chamilo's forum** ("Wavelet project")—first-come, first-served.
- Submit your assigments on TEIDE

Deadlines

Instruction	Deadline
Form pairs on TEIDE	October 5
Anaconda installation	October 5
Lab session 1 (06/10)	October 12
Article choice and summary	November 9
Lab session 2 (10/11)	November 16
Lab session 3 (15/12)	–
Project (Lab session 4 on 05/01)	January 31

Mark scheme

1. **The lab sessions** are each one graded out of 2,5 points

2. **The project** is graded out of 15 points
   - Choice of the article (difficulty, length, etc.): *1 point*
   - Summary and outline (aligned with the stated objectives): *2 points*
   - Report writing (including explanation of the method, novelty of the paper, etc.): *3 points*
   - Codes (either from scratch or using existing libraries): *4 points*
   - Numerical results (replication or extension of results): *3 points*
   - Interpretations of the results: *2 points*

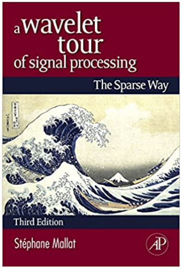
Penalties:

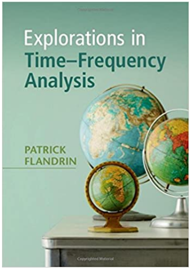
- Students **without a partner**: -2 points
- **Plagiarism** (copying codes from web or other students): mark / 2
- **Late submission**: -10% per day

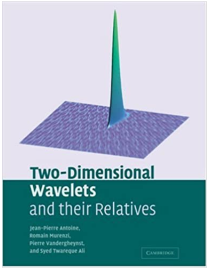
Course materials

Books

- **S. Mallat**, *A wavelet tour of signal processing*, Academic press, third edition, 2009.
- **P. Flandrin**, *Explorations in Time-Frequency Analysis*, Cambridge University Press, Cambridge (UK), 2018.
- **J-P. Antoine**, R. Murenzi, P. Vandergheynst and S.T. Ali, *Two-dimensional Wavelets and Their Relatives*, Cambridge University Press, Cambridge (UK), 2004.







Course materials

Links

- **WaveLab** (free Matlab toolbox)

  https://github.com/gregfreeman/wavelab850

- **A numerical tour of Signal/Image Processing** (by Gabriel Peyré)

  http://www.numerical-tours.com/

- **PyWavelets** (python)

  https://pywavelets.readthedocs.io/

- **Chamilo's page** (containing this course, lab sessions, forum, etc.)

  https://chamilo.grenoble-inp.fr/courses/ENSIMAGWMM9AM50/

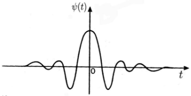
Course Wavelets and Applications – Outline

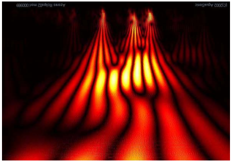
1. From Fourier to the 1D Continuous Wavelet Transform
2. Wavelet zoom: a local characterization of functions
3. Lab 1: 1D Continuous Wavelet Transform
4. The 2D Continuous Wavelet Transform
5. The 1D Discrete Wavelet Transform and Multi-resolution Analysis
6. The 2D Discrete Wavelet Transform and Multi-resolution Analysis
7. Approximation in wavelet bases (sparsity, compression, denoising)
8. Lab 2: Fast Wavelet Transform, image compression and denoising
9. The Laplacian of a graph and its applications
10. The graph Fourier transform and wavelets on graphs
11. Lab 3: Applications of Fourier and wavelets on graphs
12. Lab 4: Dedicated to the project

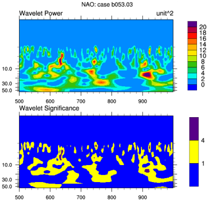
From Fourier to the 1D Continuous Wavelet Transform

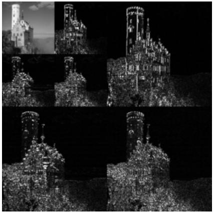
A success story

- *Wavelets for Data representation*

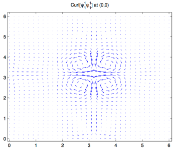




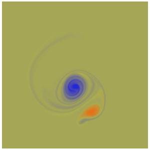




- *Wavelets for numerical simulation*



Divergence-free wavelet



Direct Simulation of Turbulence

Credits: Valérie Perrier

A success story

WSQ (1993) is the **FBI**'s **Wavelet** Scalar Quantization: it is a national standard for the collecting, encoding, storing, and retrieving digitized fingerprint images.

JPEG 2000

JPEG 2000 is an image coding system that uses state-of-the-art compression techniques based on **wavelet technology**.

Academy Sci-Tech Award 2013

Awarded to **Theodore Kim**, **Nils Thuerey**, **Dr. Markus Gross** and **Doug James** for the invention, publication and dissemination of "**Wavelet Turbulence**" software.

Credits: Valérie Perrier

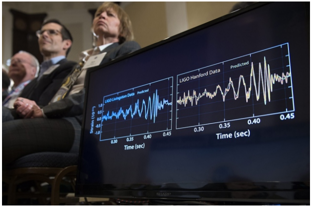
A success story

- **Abel Prize (2017)** : **Yves Meyer**, for his pivotal role in the development of the mathematical theory of wavelets.

" **Wavelet analysis** *has been applied in a wide variety of arenas as diverse as applied and computational harmonic analysis, data compression, noise reduction, medical imaging, archiving, digital cinema, deconvolution of the Hubble space telescope images, and the recent LIGO detection of gravitational waves created by the collision of two black holes."*
[http://www.abelprize.no/]

The Abel Lecture (Yves Meyer)

www.youtube.com/watch?v=wxmzHwd3z34



Wavelet theory helped LIGO to detect gravitational waves.

Credits: Valérie Perrier

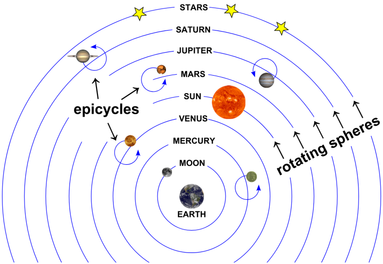
Back to the time of Ptolemy...

Who Wants to Be a Millionaire?

A pre-Copernican TV show

Credits: "Qui veut gagner des millions" (http://www.youtube.com/watch?v=ekmtqODjrSI)

Why was Ptolemy's system so efficient?



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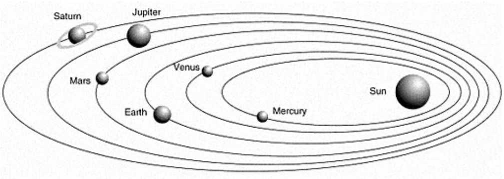


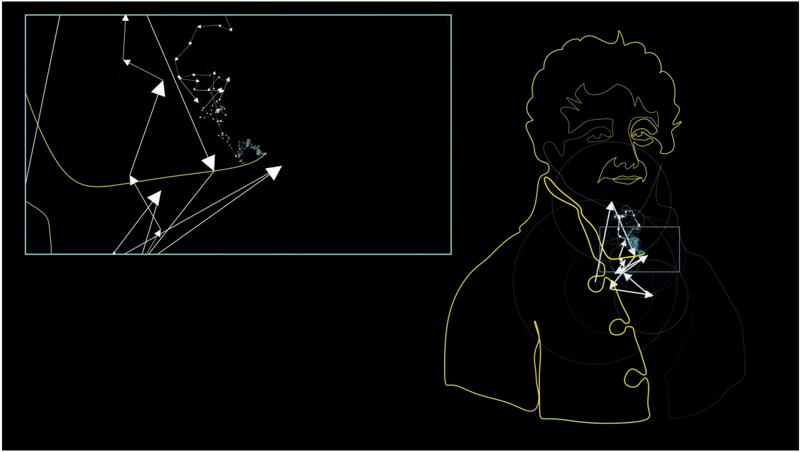
Credits: Joshua Hershey & Universe Today

Because of that...

Credits: Carman & Serra (http://www.youtube.com/watch?v=QVuU2YCwHjw)

... or more precisely thanks to him: Joseph Fourier

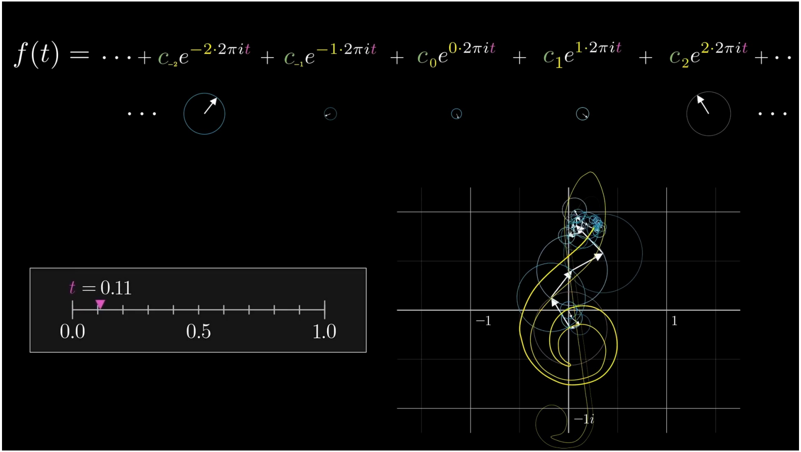
Just as Mr. Jourdain spoke prose without realizing it, astronomers used Fourier series unknowingly



Credits: 3blue1brown (http://www.youtube.com/watch?v=-qgreAUpPwM)

Fourier series: an intuition behind the decomposition

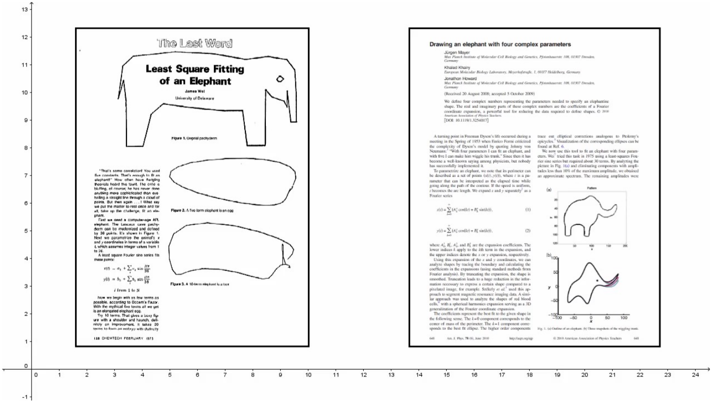
Periodic signals can be decomposed onto the Fourier basis



Credits: 3blue1brown (http://www.youtube.com/watch?v=r6sGWTCMz2k)

Draw me a (light weight) elephant

The fewer parameters the better

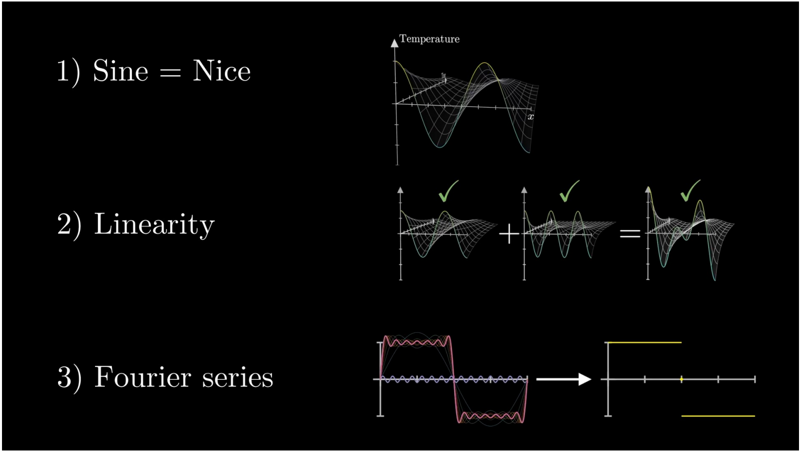


> *With four parameters I can fit an elephant, and with five I can make him wiggle his trunk.*
>
> **John von Neumann**

Credits: El Jj (http://www.youtube.com/watch?v=uazPP0ny3XQ)

Fourier series: is it only useful for drawing?

How Joseph Fourier solved the heat equation



Credits: 3blue1brown (http://www.youtube.com/watch?v=r6sGWTCMz2k)

Fourier series analysis

The **Fourier analysis** decomposes a signal (function) $f(x)$ ($x =$*times*) into a sum of sinusoidal functions:

- For a T-periodic function f, with $f \in L^2(0, T)$:

$$f(x) = \sum_{n\in\mathbb{Z}} c_n(f) \mathrm{e}^{2i\pi \frac{n}{T} x} \quad \text{(synthesis)}$$

where the Fourier coefficients are:

$$c_n(f) = \frac{1}{T} \int_0^T f(x) \mathrm{e}^{-2i\pi \frac{n}{T} x} \, \mathrm{d}x \quad \text{(analysis)}$$

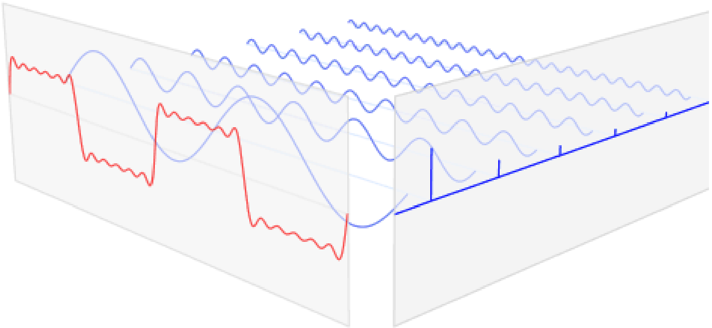
related to the *frequency* $\frac{n}{T}$ (in Hz).

Parseval equality:

$$\sum_{n\in\mathbb{Z}} |c_n(f)|^2 = \frac{1}{T} \int_0^T |f(x)|^2 \mathrm{d}x \quad \text{(energy conservation)}$$

Fourier series limitations

Discontinuities require a lot of sinusoids to be described



$$f(x) = \begin{cases} -1 \text{ if } -\pi \leqslant x < 0 \\ +1 \text{ if } 0 \leqslant x < \pi \end{cases} = \sum_{n=1}^{+\infty} \frac{4}{\pi(2n-1)} \sin((2n-1)x)$$

Credits: Wikipedia (https://en.wikipedia.org/wiki/Fourier_series)

Fourier transform

- For a function $f \in L^2(\mathbb{R})$:

$$f(x) = \int_{-\infty}^{+\infty} \hat{f}(\nu) \mathrm{e}^{2i\pi\nu x} \, \mathrm{d}\nu \quad \text{(synthesis)}$$

where the Fourier transform of f is:

$$\hat{f}(\nu) = \int_{-\infty}^{+\infty} f(x) \mathrm{e}^{-2i\pi\nu x} \, \mathrm{d}x \quad \text{(analysis)}$$

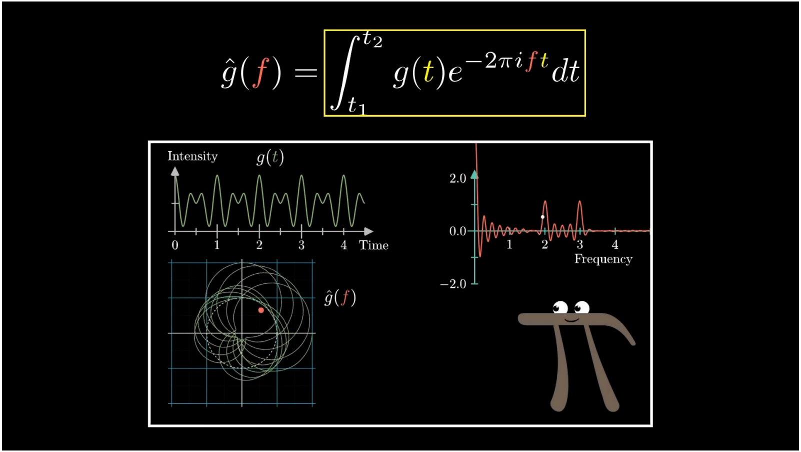
gives information on f for the frequency ν.

Plancherel-Parseval equality:

$$\int_{-\infty}^{+\infty} |\hat{f}(\nu)|^2 \mathrm{d}\nu = \int_{-\infty}^{+\infty} |f(x)|^2 \mathrm{d}x \quad \text{(energy conservation)}$$

Fourier transform: an intuition behind the transformation

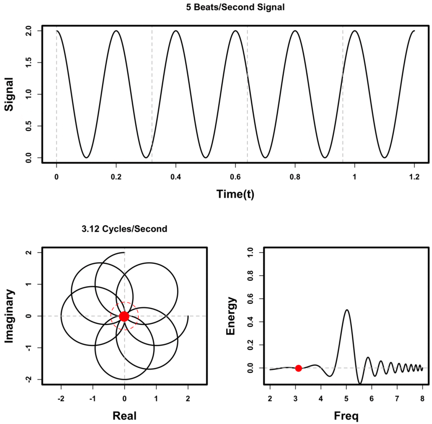
aperiodic signals can also be decomposed onto the continuous dictionary of exponentials



Credits: 3blue1brown (https://www.youtube.com/watch?v=spUNpyF58BY)

Visualization of the Fourier Transform

Wrap the signal around a circle



$$\hat{g}(f) = \frac{1}{N}\sum_{k=1}^{N} g(t_k)e^{-2\pi i f t_k}$$

To find **the energy** at a particular frequency, **the signal** is **wrapped around a circle** at the particular frequency and **the points along the path are averaged.**

Credits: Elan Ness-Cohn

Limitations of the Fourier Transform

Example: Two musical notes played simultaneously

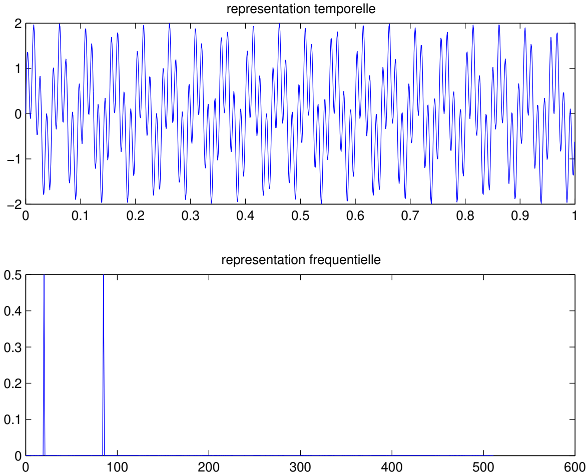
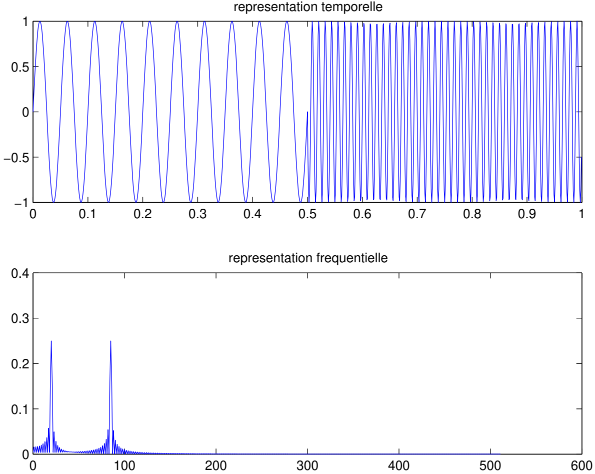


Figure: Signal $f(x) = \sin(40\pi x) + \sin(170\pi x)$ (top), and modulus of its Fourier Transform $\widehat{f}(\nu)$ (buttom)

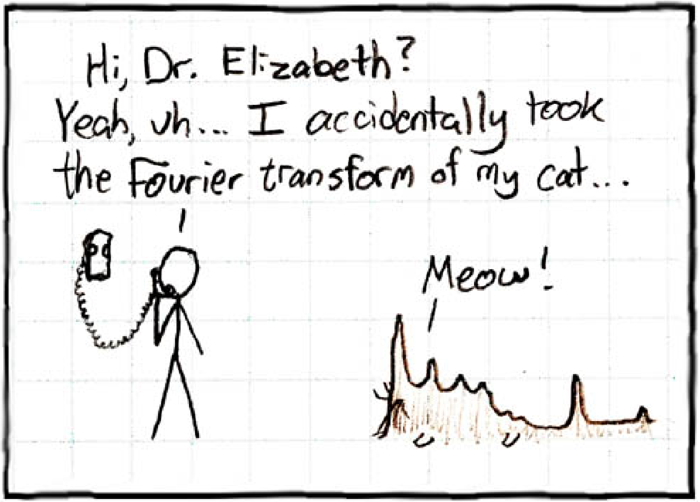
Fourier transform limitations

Example: Two musical notes played sequentially

The frequency analysis does not provide information about transient phenomena in the signal, leading to a loss of temporal localization



Fourier cat transformation



Credits: xkcd ♯26

Take home message

To sum up

- Periodic functions (such as the motion of planets along closed orbits) can be approximated by epicycles, which corresponds to Fourier series.
- Fitting data does not necessarily imply that the underlying mechanics are understood; overly complex models can lead to overfitting. The question remains: what constitutes a good model or theory?
- The relativity of motion allows us to consider different coordinate systems for describing trajectories. This principle, well-known among physicists, shows that the choice of reference frame can significantly simplify mathematical calculations.
- An appropriate representation of the signal can also reduce the number of parameters needed to encode its information.

$\Rightarrow$ **Toward a sparse representation of signals**

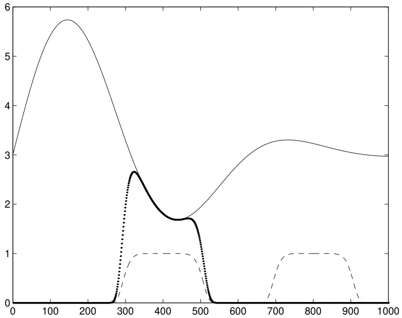
Take home message

To sum up

- Fourier series decomposition allowed Joseph Fourier to solve partial differential equations (such as the heat equation).
- The extension to the Fourier transform for aperiodic signals reveals the frequency content of the signal, but it suffers from similar issues:
- Discontinuities result in many significant coefficients in the decomposition, with decreasing amplitudes encoding the overall regularity of the signal.
- The Fourier transform lacks temporal localization, making it incapable of capturing transient phenomena in the signal.

$\Rightarrow$ **Toward a time-frequency representation of signals**

Short Time Fourier Transform



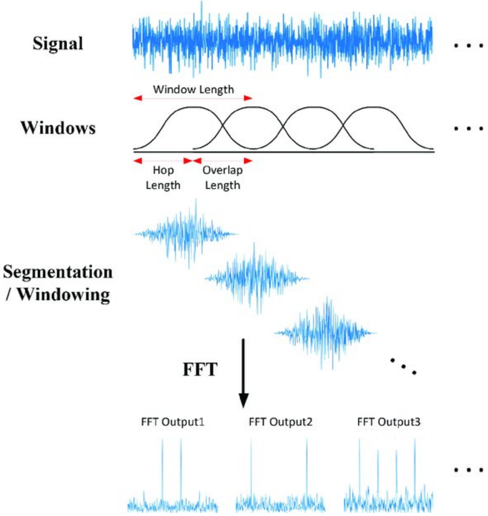
Consider multiplying the signal f by a **real window function** $x \mapsto w(x - b)$ of size a_0 centered at time b. The computation of the Fourier transform of this product allows us to analyze the signal in a localized manner in both time and frequency.

The Short-Time Fourier Transform (STFT) of the signal is defined as:

$$Sf(\nu, b) = \int_{-\infty}^{+\infty} f(x)w(x-b)\, \mathrm{e}^{-2i\pi\nu x}\, \mathrm{d}x$$

where b represents *time* and ν *frequency*.

Short Time Fourier Transform

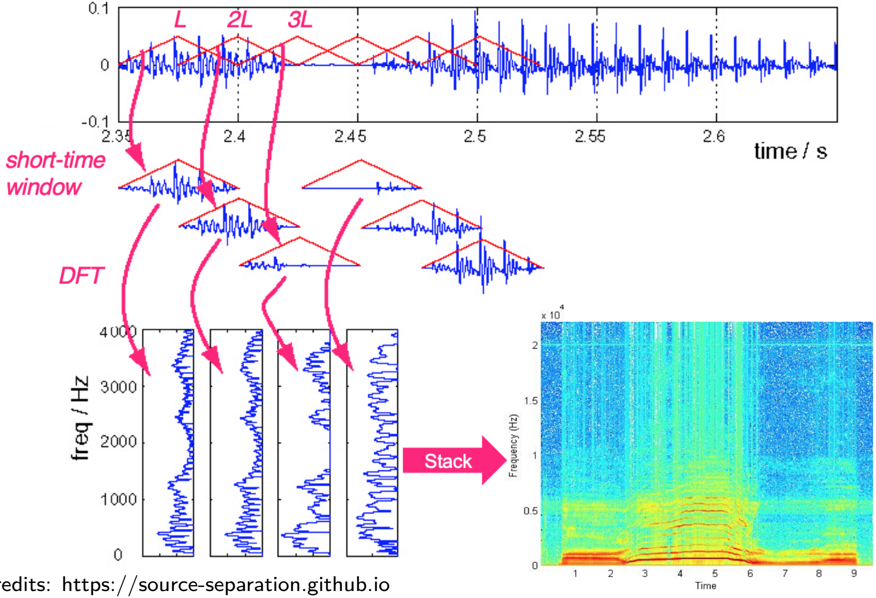


- This transform provides a redundant **time-frequency representation** of the signal.
- The original signal f can be recovered from its STFT coefficients using the **inverse Short Time Fourier Transform**:

$$f(x) = C_h \iint_{\mathbb{R}^2} Sf(\nu, b) w(x - b) \mathrm{e}^{2i\pi\nu x} \, \mathrm{d}\nu \, \mathrm{d}b$$

Credits: Jeon et al. (2020)

From the STFT to the Spectrogram $|Sf(\nu, b)|^2$



Credits: https://source-separation.github.io

Special case: the Gabor Transform

- In the Short Time Fourier Transform

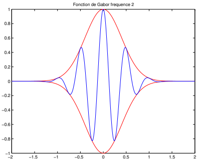
$$Sf(\nu, b) = \int_{-\infty}^{+\infty} f(x) \boxed{w(x-b)\mathrm{e}^{-2i\pi\nu x}} \,\mathrm{d}x = \langle f, \psi_{\nu,b} \rangle$$

the analyzing functions are:

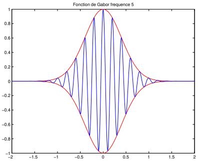
$$\psi_{\nu,b} = w(x-b)\mathrm{e}^{2i\pi\nu x}$$

- In the **Gabor transform** (1946) the window w is a Gaussian of scale σ: $w(x) = \frac{1}{\sigma}\mathrm{e}^{-\pi(\frac{x}{\sigma})^2}$ and the Gabor functions are then ($\sigma = 1$):

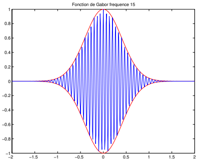
$$g_{\nu,b} = \mathrm{e}^{-\pi(x-b)^2}\mathrm{e}^{2i\pi\nu x}$$



(a) $\nu = 2$



(b) $\nu = 5$



(c) $\nu = 15$

Short Time Fourier Transform

Example of two musical notes

The time-frequency analysis allows us to recover both the frequencies (notes) and the temporal information (order in time) of the signal:

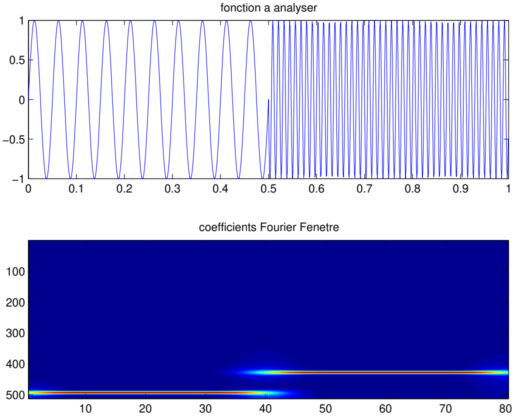
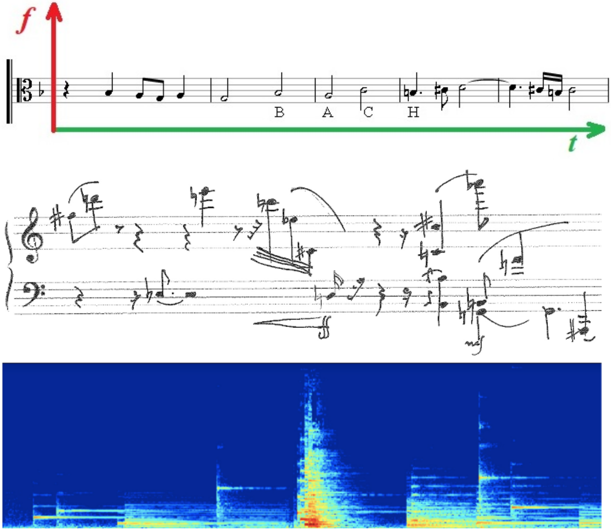


Figure: Time-frequency plane with b on the x-axis and ν on the y-axis, representing the density energy $|Sf(\nu, b)|^2 = |\langle f, g_{\nu,b}\rangle|^2$ called the spectrogram.

Short Time Fourier Transform

Analogy with music scores: an example with a piano



Credits: Patrick Flandrin, "Au-delà de Fourier, un monde qui vibre" (interstices.info)

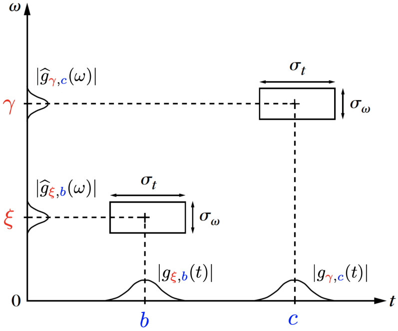
Heisenberg boxes

Time-frequency localization and spread

$$g_{\xi,b}(t) = w(t-b)e^{i\xi t} \longleftrightarrow \widehat{g}_{\xi,b}(\omega) = \widehat{w}(\omega-\xi)e^{-ib(\omega-\xi)}$$

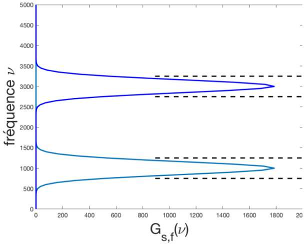
$$\sigma_t^2 = \int_{-\infty}^{\infty}(t-b)^2|g_{\xi,b}(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} t^2|w(t)|^2 \mathrm{d}t$$

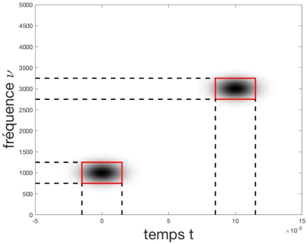
$$\sigma_\omega^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\omega-\xi)^2|\widehat{g}_{\xi,b}(\omega)|^2 \mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2|\widehat{w}(t)|^2 \mathrm{d}\omega$$



Heisenberg boxes

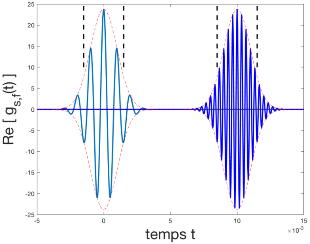
Example: Gabor limits





$$g_{s,f}(t) = w(t-s)e^{j2\pi ft}$$

$$w(t) = (\pi\sigma^2)^{-1/4}e^{-t^2/2\sigma^2}$$



Credits: Pierre Chainais, "De la transformée de Fourier à l'analyse temps-fréquence bivariée"

Heisenberg boxes

Time-frequency localization and spread

- Can we construct a function f with energy that is highly localized in time, while its Fourier transform $\widehat{f}$ has energy concentrated in a small frequency interval?
- To reduce the time spread of f, we can scale it by $a < 1$, while keeping its total energy constant:

$$f_a(t) = \frac{1}{\sqrt{a}} f\left(\frac{t}{a}\right), \quad \|f_a\|^2 = \|f\|^2$$

- The corresponding Fourier transform is dilated by a factor of $1/a$:

$$\widehat{f}_a(\omega) = \sqrt{a}\,\widehat{f}(a\omega)$$

$\Rightarrow$ Thus, we lose frequency localization for what we gain in time localization. Underlying this is a trade-off between time and frequency localization.

Credits: S. Mallat (Wavelet tour)

Heisenberg's indeterminacy relations

Defining the average location and frequency respectively by:

$$b = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} t|f(t)|^2 \mathrm{d}t, \quad \xi = \frac{1}{2\pi\|f\|^2} \int_{-\infty}^{\infty} \omega|\widehat{f}(\omega)|^2 \mathrm{d}\omega$$

The variances around these average values are respectively:

$$\sigma_t^2 = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} (t-b)^2|f(t)|^2 \mathrm{d}t, \quad \sigma_\omega^2 = \frac{1}{2\pi\|f\|^2} \int_{-\infty}^{\infty} (\omega-\xi)^2|\widehat{f}(\omega)|^2 \mathrm{d}\omega$$

Theorem (Heisenberg's indeterminacy relations)

The temporal variance and the frequency variance of $f \in L^2(\mathbb{R})$ satisfy

$$\boxed{\sigma_t \, \sigma_\omega \geqslant \frac{1}{2}}$$

This inequality is an equality iff $\exists (b, \xi, c_1, c_2) \in \mathbb{R}^2 \times \mathbb{C}^2$ such that

$$f(t) = c_1 \mathrm{e}^{i\xi t - c_2(t-b)^2}$$

Credits: S. Mallat

Heisenberg's indeterminacy relations

Proof *(Weyl)*: this proof supposes that $\lim_{|t|\to+\infty} \sqrt{t} f(t) = 0$ $(*)$ but the theorem is valid for any $f \in L^2(\mathbb{R})$. The average time and frequency location of $\mathrm{e}^{-i\xi t} f(t+b)$ is zero. Thus, it is sufficient to prove the theorem for $b = \xi = 0$. Since $\widehat{f'(t)}(\omega) = i\omega \widehat{f}(\omega)$, the Plancherel identity applied to $i\omega\widehat{f}(\omega)$ yields

$$\sigma_t^2 \sigma_\omega^2 = \frac{1}{\|f\|^4} \left(\int_{-\infty}^{\infty} |t\, f(t)|^2 \mathrm{d}t \right) \left(\int_{-\infty}^{\infty} |f'(t)|^2 \mathrm{d}t \right) \quad (**)$$

Schwarz's inequality and the assumption $(*)$ [for the last equality] imply

$$\sigma_t^2 \sigma_\omega^2 \geqslant \frac{1}{\|f\|^4} \left(\int_{-\infty}^{\infty} |t\, f'(t) f^*(t)| \mathrm{d}t \right)^2 \qquad \forall z \in \mathbb{C},\ |z| \geqslant \mathrm{Re}(z) = \frac{z + z^*}{2}$$

$$\geqslant \frac{1}{\|f\|^4} \left(\int_{-\infty}^{\infty} \frac{t}{2} (f'(t) f^*(t) + f'^*(t) f(t))\, \mathrm{d}t \right)^2$$

$$\geqslant \frac{1}{4\|f\|^4} \left(\int_{-\infty}^{\infty} t (|f(t)|^2)'\, \mathrm{d}t \right)^2 \overset{\text{IBPF}}{=} \frac{1}{4\|f\|^4} \left(\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t \right)^2 = \frac{1}{4}$$

Credits: S. Mallat (Wavelet tour) (IBPF = Integration By Parts Formula)

Heisenberg's indeterminacy relations

Proof: To obtain an equality, Schwarz's inesquality applied to $(**)$ must be an equality. This implies that there exists $c_2 \in \mathbb{C}$ such that

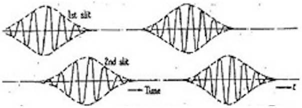
$$f'(t) = -2c_1 t f(t)$$

Thus, there exists $c_1 \in \mathbb{C}$ such that

$$f(t) = c_1 \mathrm{e}^{-c_2 t^2}$$

When $b \neq 0$ and $\xi \neq 0$ a time and frequency translation yield the result.

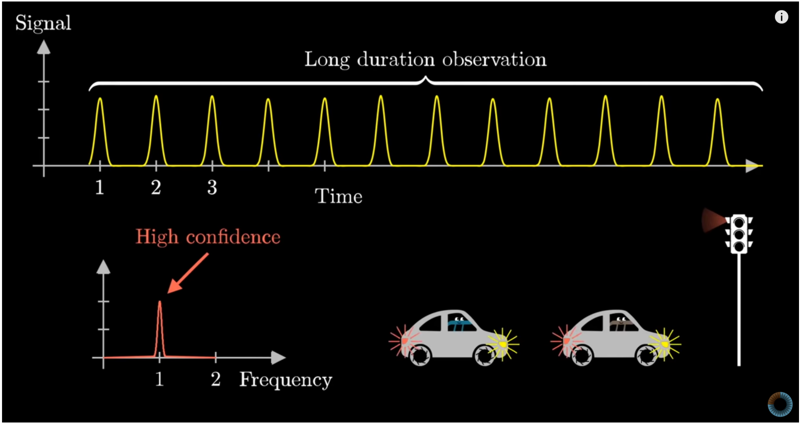
Remark: motivated by quantum mechanics, Gabor proposed time-frequency atoms that have a minimal spread in a time-frequency plane. By showing that signal decompositions over the dictionary of Gabor atoms are closely related to our perception of sounds, and that they exhibit important structures in speech and music recordings, he demonstrated the importance of localized time-frequency signal processing.



Credits: S. Mallat (Wavelet tour)

Heisenberg's indeterminacy relations

Some intuitions behind



$$\mathrm{TF}(f \cdot \Pi_{[-a/2,a/2]}) = \mathrm{TF}(f) * \mathrm{TF}(\Pi_{[-a/2,a/2]}) = \mathrm{TF}(f) * a\mathrm{sinc}(\pi a \cdot)$$

Credits: 3blue1brown (http://www.youtube.com/watch?v=MBnnXbOM5S4)

Heisenberg's indeterminacy relations

Some intuitions behind

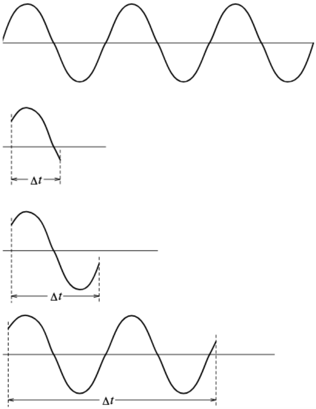


Figure: Improved frequency measurement over longer time intervals. The uncertainty in the frequency Δf decreases as the measurement interval Δt increases, and vice versa.

Credits: Bruce MacLennan (Gabor Representation)

Heisenberg's indeterminacy relations

Some intuitions behind

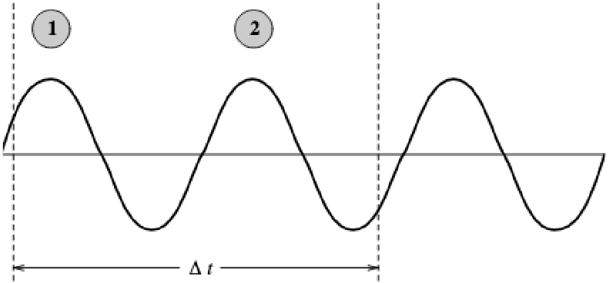


Figure: Measuring frequency by counting maxima in a given time interval. The circled numbers indicate the maxima counted during the measurement interval Δt. Since signals of other frequencies could also have the same number of maxima in that interval, there is an uncertainty Δf in the frequency.

Credits: Bruce MacLennan (Gabor Representation)

Heisenberg's indeterminacy relations

Some intuitions behind

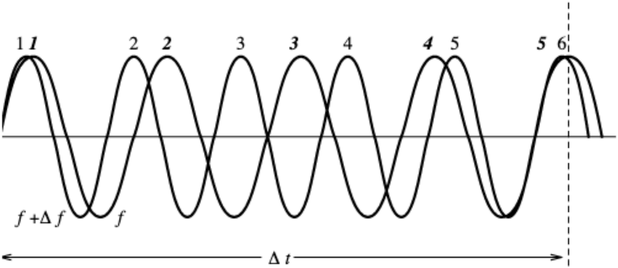


Figure: Minimum time interval Δt to detect frequency difference Δf. If two signals differ in frequency by Δf, then a measurement of duration $\Delta t \geqslant 1/\Delta f$ is required to guarantee a difference in counts of maxima. (Italic numbers indicate maxima of signal of frequency f, roman numbers indicate maxima of signal of higher frequency $f + \Delta f$)

$$(f + \Delta f)\Delta t - f\Delta t \geqslant 1 \;\Leftrightarrow \Delta f\,\Delta t \geqslant 1$$

Credits: Bruce MacLennan (Gabor Representation)

Short Time Fourier Transform

Examples

1. A sinusoidal wave $f(t) = e^{i\xi_0 t}$ whose Fourier transform is a Dirac $\widehat{f}(\omega) = 2\pi\delta(\omega - \xi_0)$ has a STFT:

$$Sf(\xi, b) = \widehat{w}(\xi - \xi_0)e^{-ib(\xi-\xi_0)}$$

   Its energy is spread over the frequency interval

$$\xi \in [\xi_0 - \sigma_\omega/2, \xi_0 + \sigma_\omega/2]$$

2. A Dirac $f(t) = \delta(t - b_0)$ has a STFT:

$$Sf(\xi, b) = w(b - b_0)e^{-i\xi b_0}$$

   Its energy is spread in the time interval

$$b \in [b_0 - \sigma_t/2, b_0 + \sigma_t/2]$$

Limitation of the Short Time Fourier Transform

The STFT cannot separate events of a distance smaller than a_0, that is to localize the two frequencies and the transient phenomena.

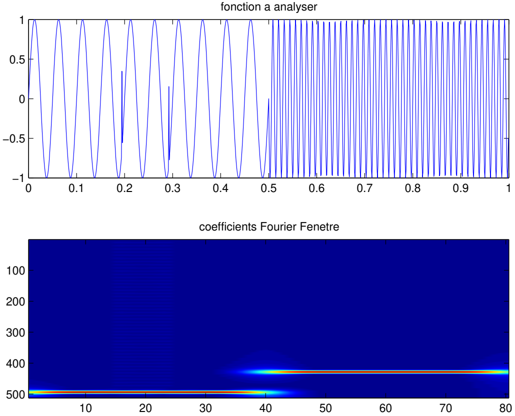


Figure: Signal $f_2 = f_1 + \delta_1 + \delta_2$ and its Gabor transform with $a_0 = 0.05$

Limitation of the Short Time Fourier Transform

The STFT cannot separate events of a distance smaller than a_0, that is to localize the two frequencies and the transient phenomena.

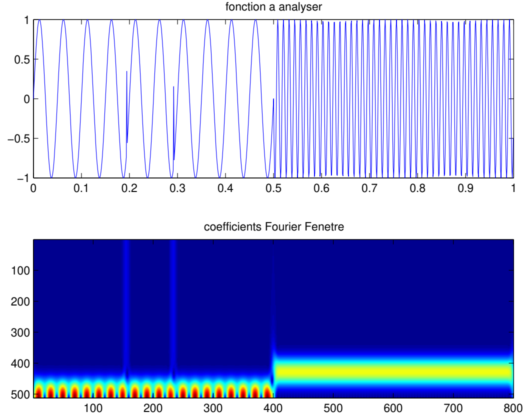
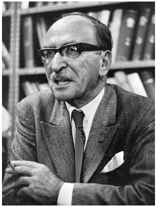


Figure: Signal $f_2 = f_1 + \delta_1 + \delta_2$ and its Gabor transform with $a_0 = 0.005$

Pioneer works on wavelets

- **Jean Morlet** research engineer at ELF Aquitaine discovered wavelets for solving signal processing problems arising from oil exploration.
- **Alex Grossmann** recognized in the Morlet wavelets something similar to coherent states formalism in quantum mechanics and developed an exact inversion formula for the wavelet transform.
- They developed the mathematics of the continuous wavelet transforms in their article: "Decomposition of Hardy Functions into Square Integrable Wavelets of Constant Shape" (1984)



Dennis Gabor

Alex Grossmann

Jean Morlet

"Gaborettes" vs Morlet wavelets

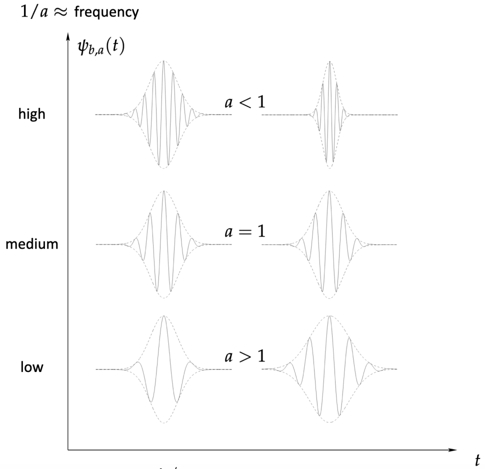
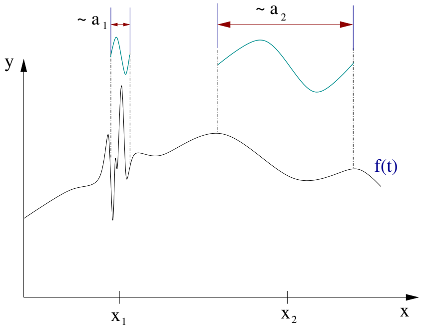


Figure: (Left) Gabor $\psi_{a,b}(t) = \mathrm{e}^{it/a}\psi(t-b)$, (right) Morlet $\psi_{a,b}(t) = a^{-1/2}\psi\left(\frac{t-b}{a}\right)$

Gabor $\Rightarrow$ frequency modulation within a constant window width

Wavelets $\Rightarrow$ shape of $\psi_{a,b}$ doesn't change, simply dilated or compressed

The Continuous Wavelet Transform (CWT) – Definition



$$Wf(a,b) = \int_{-\infty}^{+\infty} f(x)\overline{\psi_{a,b}(x)}\,\mathrm{d}x = \langle f, \psi_{a,b} \rangle, \quad a > 0,\ b \in \mathbb{R}$$

The analyzing functions or **wavelets** are defined by:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)$$

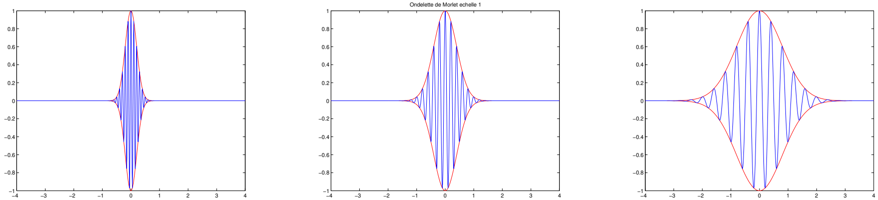
Wavelet family in physical space

Example: the Morlet wavelets

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)$$

with mother wavelet

$$\psi(x) = \cos(x)\mathrm{e}^{-10\pi x^2}$$



(a) $a = 1/2$ (b) $a = 1$ (c) $a = 2$

Figure: Morlet wavelets of scale: $a = 1/2,\ 1,\ 2$ (real part). The scale a gives the support size (inverse of a frequency), whereas b gives the position.

Wavelet analysis of the toy signal with Morlet wavelets

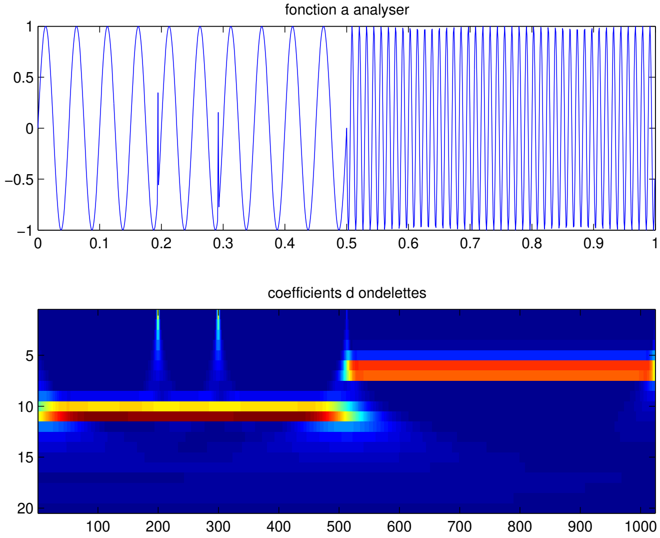


Figure: Signal f_2 (two notes + scratch) and its CWT

Wavelet definition

A function $\psi(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is a **wavelet** if it satisfies the following *admissiblity condition*:

$$C_\psi = \int_{-\infty}^{+\infty} \frac{\left|\widehat{\psi}(\nu)\right|^2}{|\nu|} \mathrm{d}\nu \; < \; \infty$$

which implies $\int_{-\infty}^{+\infty} \psi(x) \, \mathrm{d}x = 0$ (and this is equivalent if $x\psi$ integrable).

Examples

1. The (complex) Morlet wavelet
   - Mother wavelet: $\psi(x) = \mathrm{e}^{-\pi x^2} \mathrm{e}^{10 i \pi x}$
   - Its Fourier Transform: $\hat{\psi}(\nu) = \mathrm{e}^{-\pi(\nu - 5)^2}$ $(\hat{\psi}(0) = \mathrm{e}^{-25\pi} \approx 7.10^{-35})$
2. Gaussian derivatives
   - Mother wavelet: $\psi_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-\pi x^2}, \quad n \geq 1$
     (for $n = 2$, ψ_2 is called the "Mexican Hat")
   - Its Fourier Transform: $\hat{\psi}_n(\nu) = (2 i \pi \nu)^n e^{-\pi \nu^2}$

Wavelet analysis : a picture is worth a thousand words

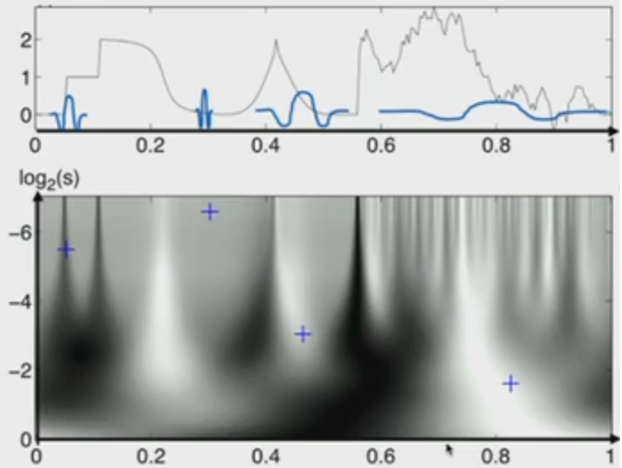


Figure: Correlations of the signal f with $\psi_{s,u}$ the Morlet wavelet dilated by s and translated by u (curves in blue on the top) whose coordinates $(u, \log_2(s))$ (represented by the blue points on the bottom) of the scalogram contain the value $\langle f, \psi_{s,u} \rangle$

Credits: S. Mallat

Fourier Transform of wavelets

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right) \longleftrightarrow \hat{\psi}_{a,b}(\nu) = \sqrt{a}\hat{\psi}(a\nu)\mathrm{e}^{-2i\pi b\nu}$$

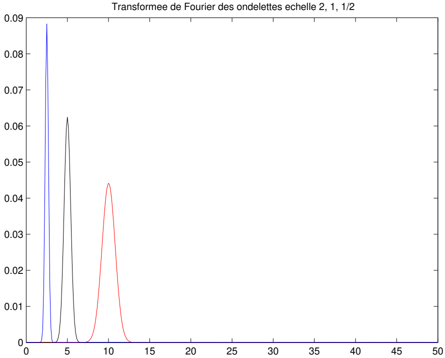


Figure: Fourier Transform (modulus) of Morlet wavelets of scales $a = 2,\ 1,\ 1/2$. Wavelets behaves as band-pass filters around frequency $\nu = \frac{\nu_0}{a}$, where ν_0 is the peak wavenumber (max of $\hat{\psi}$). For the Morlet wavelet, $\nu_0 = 5$.

Equivalent definition

Let $f \in L^2(\mathbb{R})$. Forall $a > 0$, $b \in \mathbb{R}$,

$$Wf(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} \, \mathrm{d}x$$

$$Wf(a,b) = \sqrt{a} \int_{-\infty}^{+\infty} \hat{f}(\nu) \overline{\widehat{\psi}(a\nu)} \mathrm{e}^{2i\pi\nu b} \, \mathrm{d}\nu$$

Proof: *From the Parseval formula*

$$Wf(a,b) = \langle f, \psi_{a,b} \rangle = \langle \hat{f}, \hat{\psi}_{a,b} \rangle$$

- In the time domain (x), $Wf(a,b)$ provides information on the signal f around point b in a vicinity of size $\sim a$.
- In the frequency domain (ν), $Wf(a,b)$ provides information on the signal $\hat{f}$ around frequency $\sim \frac{1}{a}$.

$\Rightarrow$ Wavelet analysis is a time-scale analysis

Time-frequency resolution of wavelets

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right) \longleftrightarrow \hat{\psi}_{a,b}(\nu) = \sqrt{a}\hat{\psi}(a\nu)\mathrm{e}^{-2i\pi b\nu}$$

We suppose that ψ is analytic, $\psi(0) = 0$ and $\eta = \frac{1}{2\pi}\int_0^\infty \omega|\widehat{\psi}(\omega)|^2 \mathrm{d}\omega$

$$\int_{-\infty}^{\infty}(t-b)^2|\psi_{a,b}(t)|^2\mathrm{d}t \overset{t\leftarrow\frac{t-b}{a}}{=} \int_{-\infty}^{\infty} a^2 t^2|\psi(t)|^2\mathrm{d}t = a^2\sigma_t^2$$

$$\frac{1}{2\pi}\int_0^\infty \left(\omega - \frac{\eta}{a}\right)^2 |\widehat{\psi}_{\xi,b}(\omega)|^2\mathrm{d}\omega = \frac{1}{2\pi a^2}\int_0^\infty (\omega-\eta)^2|\widehat{\psi}(\omega)|^2\mathrm{d}\omega = \frac{\sigma_\omega^2}{a^2}$$

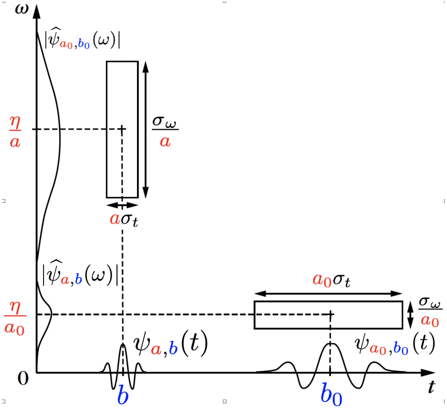
The energy spread of a wavelet time-frequency atom $\psi_{a,b}$ corresponds to a Heisenberg box centered at $(b, \xi = \eta/a)$, of size $a\,\sigma_t$ along time and σ_ω/a along frequency. The area of the rectangle remains equal to $\sigma_t\sigma_\omega$ at all scales but the resolution in time and frequency depends on a. An analytic wavelet transform defines a local time-frequency energy density

$$P_W f(b,\xi) = |Wf(a,b)|^2 = \left|Wf\left(\frac{\eta}{\xi}, b\right)\right|^2 \quad (\textit{scalogram})$$

Heisenberg boxes of two wavelets $\psi_{a,b}$ and ψ_{a_0,b_0}

$$\int_{-\infty}^{\infty} (t-b)^2 |\psi_{a,b}(t)|^2 \mathrm{d}t = a^2 \sigma_t^2$$

$$\frac{1}{2\pi} \int_0^{\infty} \left(\omega - \frac{\eta}{a}\right)^2 |\widehat{\psi}_{\xi,b}(\omega)|^2 \mathrm{d}\omega = \frac{\sigma_\omega^2}{a^2}$$



Inversion of the Continuous Wavelet Transform

Synthesis formula and energy conservation

Theorem (Calderón, Grossmann and Morlet)

Let $\psi \in L^2(\mathbb{R})$ be a real function such that

$$C_\psi = \int_{-\infty}^{+\infty} \frac{\left|\widehat{\psi}(\nu)\right|^2}{|\nu|} \, \mathrm{d}\nu \; < \; \infty$$

Any $f \in L^2(\mathbb{R})$ satisfies

$$f(x) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} Wf(a,b) \psi_{a,b}(x) \frac{\mathrm{d}a \, \mathrm{d}b}{a^2} \; (*)$$

and

$$\int_{-\infty}^{+\infty} |f(x)|^2 \mathrm{d}x = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} |Wf(a,b)|^2 \frac{\mathrm{d}a \, \mathrm{d}b}{a^2} \; (**)$$

Proof *(Synthesis formula):* For a fixed a, the CWT can be written:

$$Wf(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x)\, \overline{\psi\left(\frac{x-b}{a}\right)}\, \mathrm{d}x = (f * \check{\psi}_a)(b)$$

where we have noted:

$$\psi_a(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x}{a}\right), \quad \check{\psi}_a(x) = \psi_a(-x)$$

The right integral $b(x)$ of $(*)$ can now be rewritten as a sum of convolutions:

$$b(x) = \frac{1}{C_\psi} \int_0^{+\infty} Wf(a,\cdot) * \psi_a(x) \frac{\mathrm{d}a}{a^2} = \frac{1}{C_\psi} \int_0^{+\infty} f * \check{\psi}_a * \psi_a(x) \frac{\mathrm{d}a}{a^2}$$

$$\widehat{b}(\omega) = \frac{1}{C_\psi} \int_0^{+\infty} \widehat{f}(\omega) \sqrt{a}\, \widehat{\psi}^*(a\,\omega)\, \sqrt{a} \widehat{\psi}(a\,\omega) \frac{\mathrm{d}a}{a^2} = \frac{\widehat{f}(\omega)}{C_\psi} \int_0^{+\infty} |\widehat{\psi}(a\,\omega)|^2 \frac{\mathrm{d}a}{a}$$

By the change of variable $\xi = a\,\omega$ we get $\widehat{b}(\omega) = \frac{\widehat{f}(\omega)}{C_\psi} \int_0^{+\infty} \frac{|\widehat{\psi}(\xi)|^2}{\xi}\, \mathrm{d}\xi = \widehat{f}(\omega)$

The equality of their Fourier transform leads to $b = f$. QED □

Inversion with a different synthesis wavelet

- **Decomposition** with an *analysing wavelet* g: $a > 0$, $b \in \mathbb{R}$,

$$W_g f(a,b) = \int_{-\infty}^{+\infty} f(x) \frac{1}{\sqrt{a}} \bar{g}\left(\frac{x-b}{a}\right) \mathrm{d}x$$

- **Synthesis** with a *reconstruction wavelet* h:

$$f(x) = \frac{2}{c_{gh}} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} W_g f(a,b) \, \frac{1}{\sqrt{a}} h\left(\frac{x-b}{a}\right) \, \frac{\mathrm{d}a}{a^2} \, \mathrm{d}b$$

- **Cross-admissibility condition** on wavelets g et h ($g, h \in L^2(\mathbb{R})$):

$$c_{gh} = \int_{-\infty}^{+\infty} \frac{\bar{\hat{g}}(k) \hat{h}(k)}{|k|} \, \mathrm{d}k < +\infty$$

Remark: In this case, only h or g has to be a zero mean function.

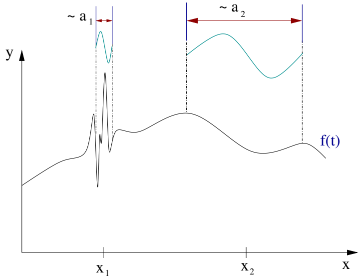
Coding – In practice

For a fixed a, the CWT is a convolution product:

$$\begin{aligned} Wf(a,b) &= \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x)\, \overline{\psi\left(\frac{x-b}{a}\right)}\, \mathrm{d}x \\ &= (f * \check{\psi}_a)(b) \end{aligned}$$

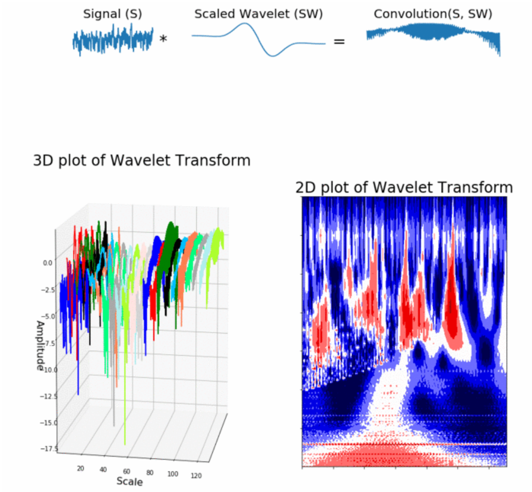
where we have noted:

$$\check{\psi}_a(x) = \frac{1}{\sqrt{a}} \psi\left(-\frac{x}{a}\right)$$



Coding – In practice

$$f * \check{\psi}_a : b \mapsto Wf(a, b)$$



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Examples and Interpretation

Example 1: pure cosine

If f is a **pure cosine** $f(x) = \cos(2\pi k x)$, then

$$\begin{aligned} Wf(a,b) &= \int_{-\infty}^{+\infty} \left(\frac{e^{2i\pi kx} + e^{-2i\pi kx}}{2} \right) \overline{\psi_{a,b}(x)} \, dx \\ &= \frac{1}{2} \left[\overline{\hat{\psi}_{a,b}(k)} + \overline{\hat{\psi}_{a,b}(-k)} \right] \\ &= \frac{\sqrt{a}}{2} \left[\overline{\hat{\psi}(ak)} e^{2i\pi kb} + \overline{\hat{\psi}(-ak)} e^{-2i\pi kb} \right] \end{aligned}$$

- If the wavelet ψ is analytic complex:

$$Wf(a,b) = \frac{\sqrt{a}}{2} \, \overline{\hat{\psi}(ak)} e^{2i\pi kb}$$

- If the wavelet ψ is real $Wf(a,b) = \sqrt{a} \, \mathrm{Re}\left(\hat{\psi}(ak) e^{-2i\pi kb} \right)$

Example 1: pure cosine

Pure cosine $f(x) = \cos(20\pi x)$ ($k = 10$)

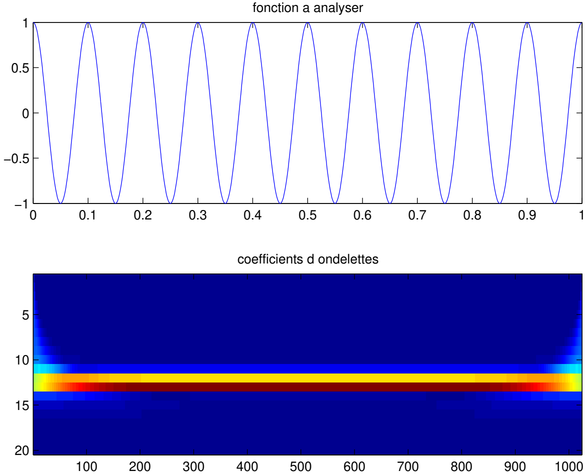


Figure: CWT (modulus), using the Morlet wavelet (analytic complex)

Example 1: pure cosine

Pure cosine $f(x) = \cos(20\pi x)$ ($k = 10$)

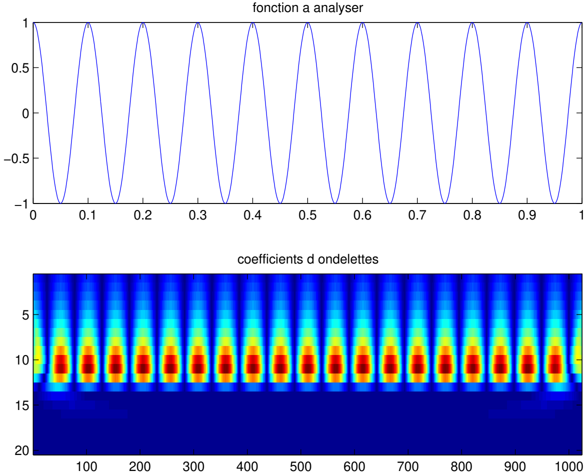


Figure: CWT (modulus), using the Gaussian derivatives (real)

Examples and Interpretation

Example 2: a Dirac

If f is a **Dirac** $f(x) = \delta(x - x_0)$ (pointwise measure supported by x_0), then:

$$
\begin{aligned}
Wf(a,b) &= \int_{-\infty}^{+\infty} \delta(x - x_0)\, \overline{\psi_{a,b}(x)}\, \mathrm{d}x \\
&= \overline{\psi_{a,b}(x_0)} \\
&= \frac{1}{\sqrt{a}}\, \overline{\psi\left(\frac{x_0 - b}{a}\right)}
\end{aligned}
$$

Remark: At each scale a, $b \to Wf(a,b)$ is the wavelet of scale a centered on x_0 (up to a symmetry).

Example 2: a Dirac

Dirac δ_{x_0}

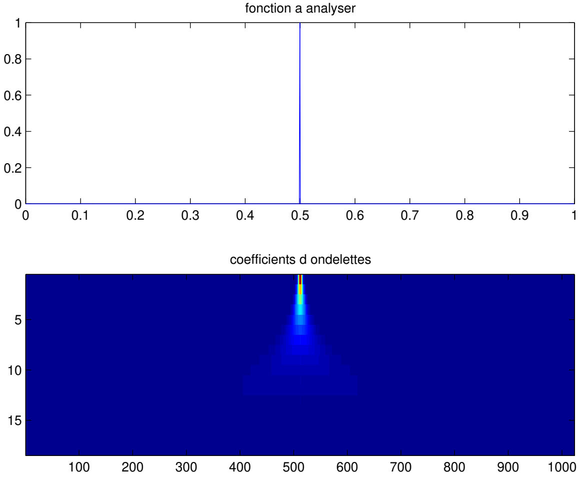


Figure: Signal "Dirac" and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)

Example 3: the periodic square wave

Periodic square wave

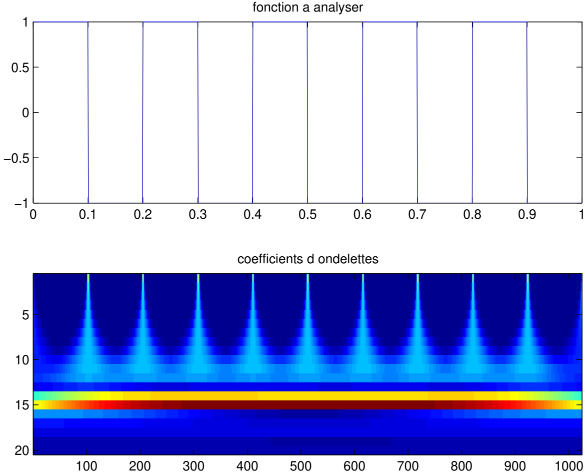


Figure: Square wave and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)

Example 4: a modulated wave

Modulated wave

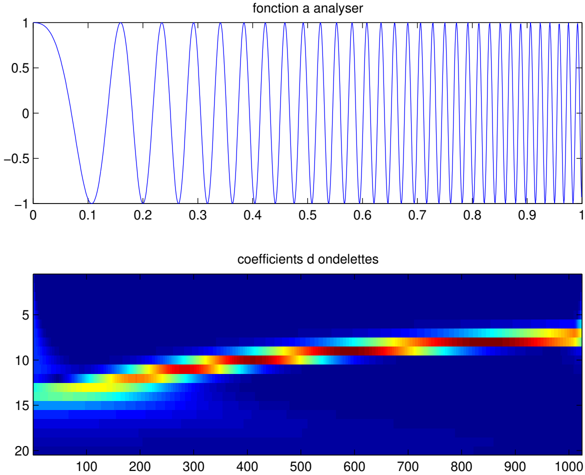


Figure: Modulated wave and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)

Example 5: 2 sinusoids with noise

2 sinusoids with noise

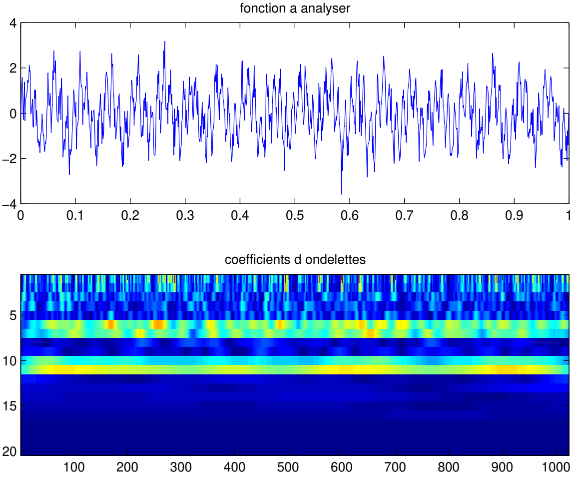


Figure: Signal and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)

Example 6: Holder function of exponant $\frac{1}{2}$

Holder function of exponant $\frac{1}{2}$

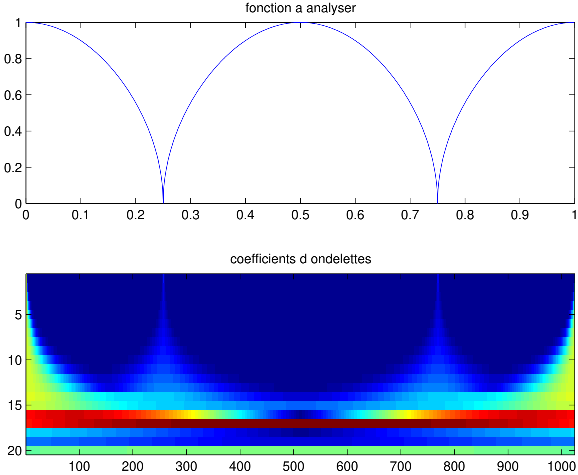
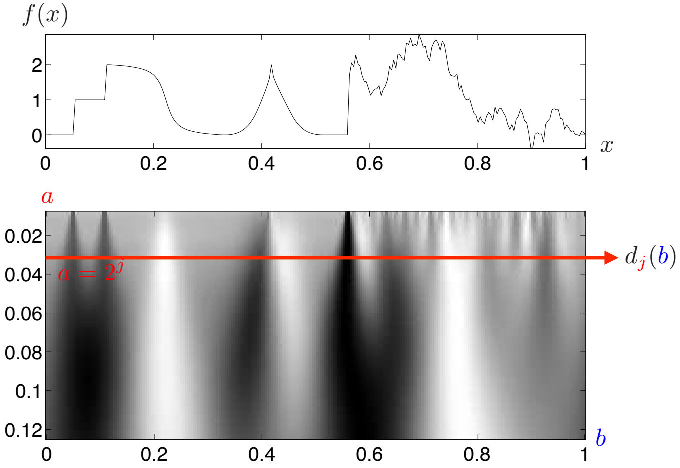


Figure: $f(x) = \sqrt{|\cos(2\pi x)|}$ and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)

The Dyadic Wavelet Transform



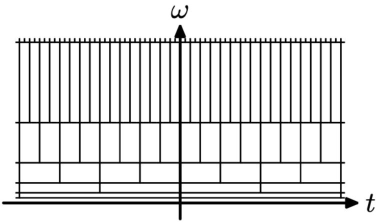
The Dyadic Wavelet Transform

For a fixed $a = 2^j$, the Dyadic Wavelet Transform is a convolution product:

$$\begin{aligned} Wf(2^j, b) &= \frac{1}{\sqrt{2^j}} \int_{-\infty}^{+\infty} f(x) \, \overline{\psi\left(\frac{x-b}{2^j}\right)} \, \mathrm{d}x \\ d_j(b) &= (f * \check{\psi}_j)(b) \end{aligned}$$

where we have noted:

$$\psi_j(x) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{x}{2^j}\right), \quad \check{\psi}_j(x) = \psi_j(-x)$$



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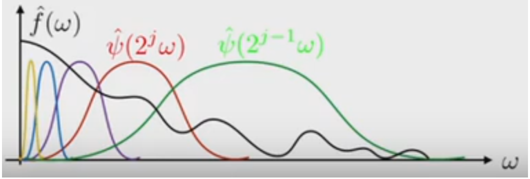
whose Fourier transform is

$$\widehat{d}_j(\omega) = \widehat{f}(\omega)\sqrt{2^j}\,\widehat{\psi}^*(2^j\omega)$$



The Dyadic Wavelet Transform

Theorem (Littlewood-Paley, 1930)

If $\sum_j |\widehat{\psi}(2^j\omega)|^2 = 1$ then

$$f(x) = \sum_j 2^{-j} \int Wf(2^j, b)\psi_{2^j,b}(x)\,\mathrm{d}b$$

Proof: Remark that

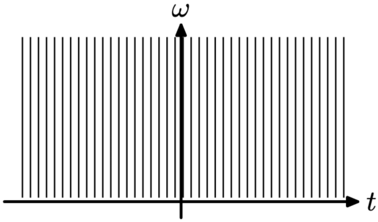
$$\int Wf(2^j, b)\psi_{2^j,b}(x)\,\mathrm{d}b = d_j * \psi_j(x)$$

then take the Fourier transform

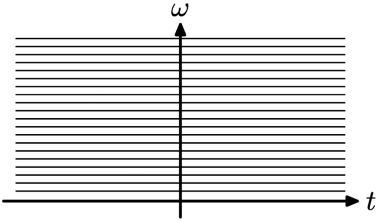
$$\begin{aligned}\sum_j 2^{-j}\widehat{d}_j(\omega)\widehat{\psi}_j(\omega) &= \sum_j 2^{-j}\widehat{f}(\omega)\sqrt{2^j}\widehat{\psi}^*(2^j\omega)\sqrt{2^j}\psi(2^j\omega) \\ &= \widehat{f}(\omega)\underbrace{\sum_j |\widehat{\psi}(2^j\omega)|^2}_{=1} = \widehat{f}(\omega)\end{aligned}$$

Take home message

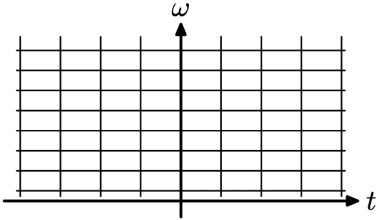
Time-frequency vs time-scale



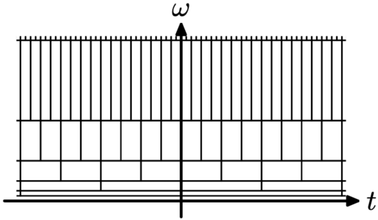
(a) Diracs



(b) Fourier



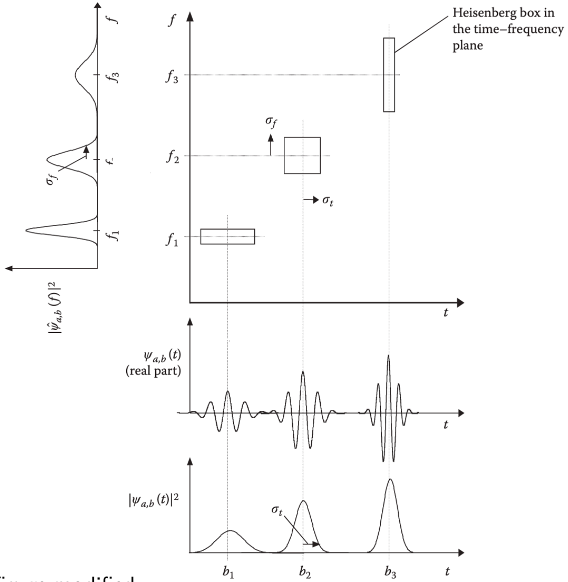
(c) STFT



(d) Wavelets dyadics

Take home message

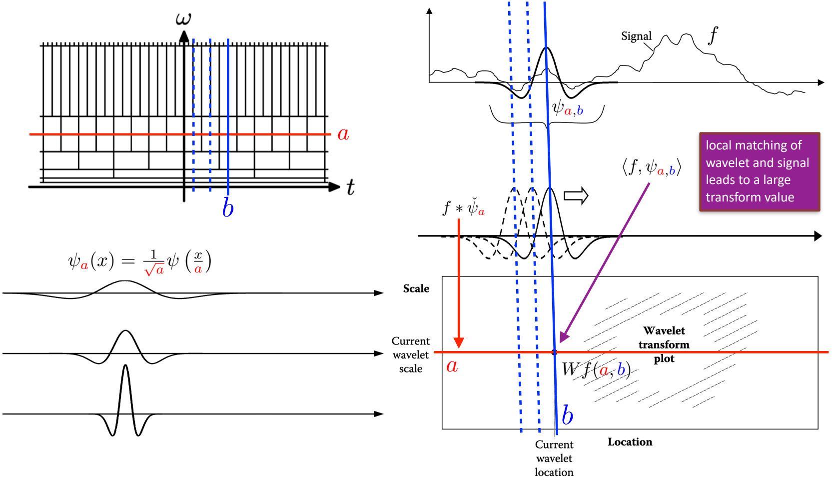
Heisenberg for wavelets



Credits: Paul S Addison's figure modified

Take home message

Scalogram construction



Credits: Paul S Addison's figure adapted