Wavelets and Applications

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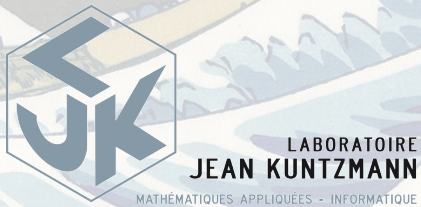


The 2D Continuous Wavelet Transform

Bidimensional Continuous Wavelet Transform

1. 2D Wavelets
2. Directional Continuous Wavelet Transform, inversion formula
3. Isotropic Wavelet Transform
4. A wavelet for image analysis: the "Canny" wavelets

2D Fourier Transform

The bidimensional Fourier transform of a function f integrable on $\mathbb{R}^2$ is defined by:

$$\hat{f}(\boldsymbol{k}) = \iint_{\mathbb{R}^2} f(\boldsymbol{x})\, \mathrm{e}^{-2i\pi \boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, \quad \forall \boldsymbol{k} \in \mathbb{R}^2$$

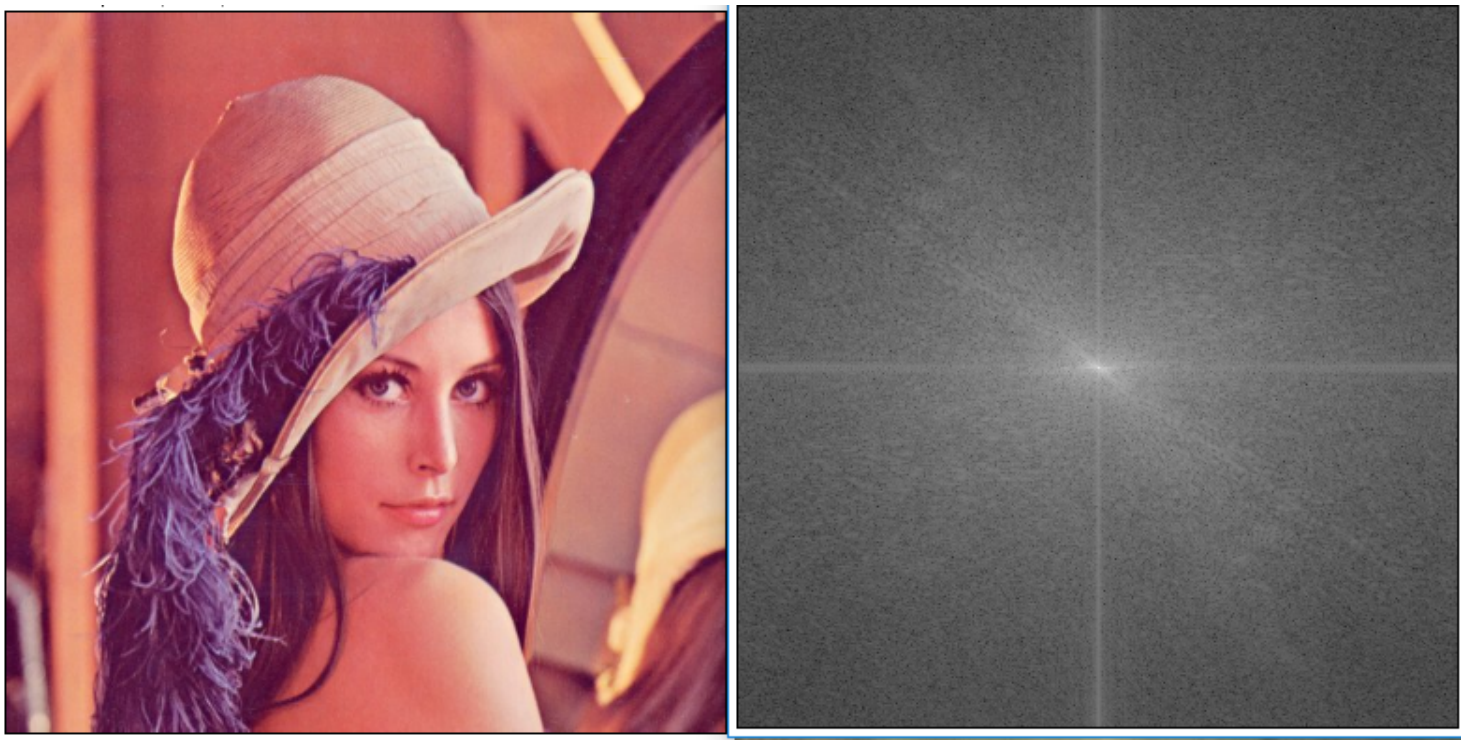
If $f \in L^2(\mathbb{R}^2)$, the inversion formula is given by:

$$f(\boldsymbol{x}) = \iint_{\mathbb{R}^2} \hat{f}(\boldsymbol{k})\, \mathrm{e}^{2i\pi \boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{k}$$

and the energy conservation writes:

$$\iint_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = \iint_{\mathbb{R}^2} |\hat{f}(\boldsymbol{k})|^2 \mathrm{d}\boldsymbol{k}$$

2D Fourier Transform



Definition of 2D wavelets

$\psi \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ is a wavelet if it satisfies the admissibility condition:

$$c_\psi = \iint_{\mathbb{R}^2} \frac{|\hat{\psi}(\boldsymbol{k})|^2}{\|\boldsymbol{k}\|^2} \, \mathrm{d}\boldsymbol{k} < +\infty$$

which implies (and is equivalent provided ψ has sufficient decay at infinity):

$$\iint_{\mathbb{R}^2} \psi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 0$$

In practice, one usually needs that ψ has p vanishing moments:

$$\iint_{\mathbb{R}^2} x_1^{\alpha_1} x_2^{\alpha_2} \psi(x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2 = 0, \quad \forall \alpha_1, \alpha_2 \in \mathbb{N} \text{ s.t. } \alpha_1 + \alpha_2 \leq p - 1$$

Remark: this means that the Fourier transform of the wavelet should behave as $\|\boldsymbol{k}\|^p$ when $\boldsymbol{k} \to 0$ in Fourier domain.

2D Wavelet family

Let $\psi(\boldsymbol{x})$ be an admissible wavelet. The wavelet family is defined by dilation, rotation, and translation from ψ:

$$\psi_{(a,\boldsymbol{b},\theta)}(\boldsymbol{x}) = \frac{1}{a}\psi\left(\mathbf{R}_{-\theta}\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)\right)$$

with $\boldsymbol{b} \in \mathbb{R}^2$ the translation parameter, a the positive scale and $\mathbf{R}^{-\theta}$ the rotation of angle θ in $\mathbb{R}^2$, corresponding to matrix

$$\mathbf{R}_{-\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Example (Anisotropic Morlet Wavelet)

Let $\boldsymbol{u} = (\cos\alpha, \sin\alpha)$ the unitary vector of direction α.
The (complex) Morlet wavelet is:

$$\psi(\boldsymbol{x}) = \mathrm{e}^{-\pi\|\boldsymbol{x}\|^2}\mathrm{e}^{10i\pi\boldsymbol{x}\cdot\boldsymbol{u}}$$

Isotropic wavelets

Example (Iterated Laplacian of Gaussian)

For $n \geq 1$, the wavelet h_{2n} is defined by:

$$h_{2n}(\boldsymbol{x}) = (-1)^n \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^n \mathrm{e}^{-\pi \|\boldsymbol{x}\|^2}$$

Its Fourier transform is given by:

$$\widehat{h}_{2n}(\boldsymbol{k}) = 4^n \pi^{2n} \, \|\boldsymbol{k}\|^{2n} \mathrm{e}^{-\pi \|\boldsymbol{k}\|^2}$$

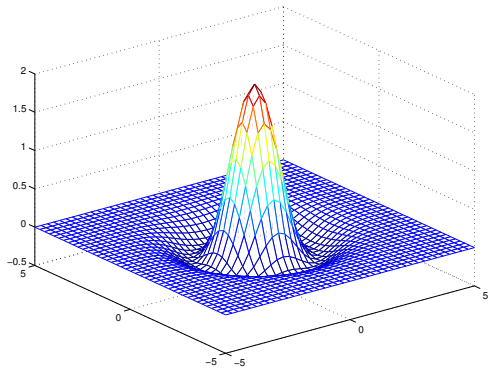
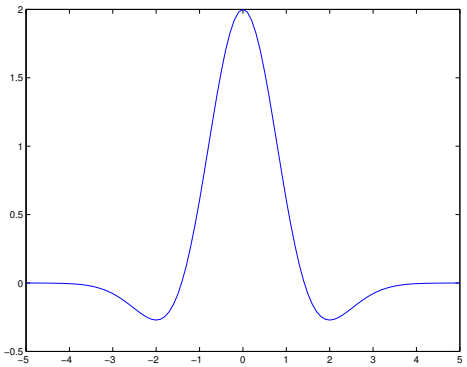
For $n = 2$, h_2 is the Laplacian of Gaussian, popular in computer vision, also called the *Mexican hat*.

Remark: the wavelet h_{2n} has exactly $2n$ vanishing moments. The maximum of its Fourier transform $\hat{h}_{2n}$ is achieved for $k_0 = \sqrt{2n}$.

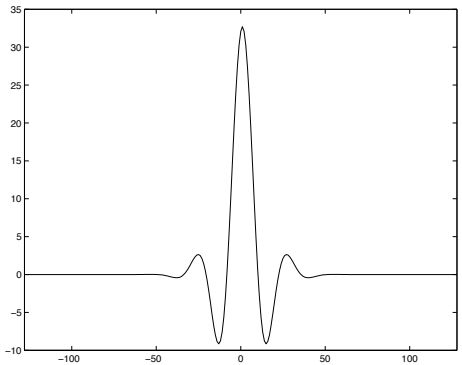
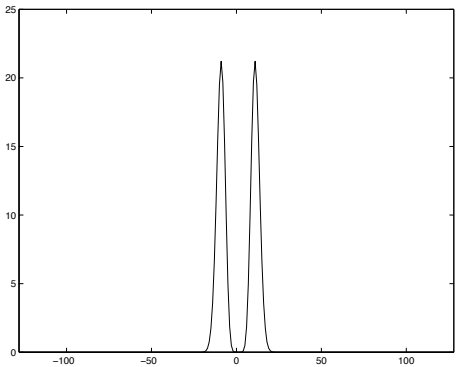
Isotropic wavelets

Example (Iterated Laplacian of Gaussian)

- Wavelet $h_2(x, y)$ (Mexican hat) and 1D section:

- 1D section in physical and Fourier space of the wavelet h_8

2D directional continuous wavelet transform

Let ψ be a 2D wavelet.

The directional wavelet transform of a given function $f \in L^2(\mathbb{R}^2)$ is defined by:

$$\begin{aligned} Wf(a, \boldsymbol{b}, \theta) &= \iint_{\mathbb{R}^2} f(\boldsymbol{x}) \, \overline{\psi_{(a,\boldsymbol{b},\theta)}(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{a} \iint_{\mathbb{R}^2} f(\boldsymbol{x}) \, \psi \left(\mathbf{R}_{-\theta} \left(\frac{\boldsymbol{x} - \boldsymbol{b}}{a} \right) \right) \mathrm{d}\boldsymbol{x} \end{aligned}$$

Applying Parseval formula, it writes:

$$Wf(a, \boldsymbol{b}, \theta) = a \iint_{\mathbb{R}^2} \widehat{f}(\boldsymbol{k}) \, \overline{\widehat{\psi}(a \, \mathbf{R}_{-\theta} \boldsymbol{k})} \mathrm{e}^{2i\pi \boldsymbol{k} \cdot \boldsymbol{b}} \, \mathrm{d}\boldsymbol{k}$$

Inversion formula

The function f can be reconstructed by:

$$f(\boldsymbol{x}) = \frac{1}{c_\psi} \int_0^{+\infty} \int_0^{2\pi} \iint_{\mathbb{R}^2} Wf(a, \boldsymbol{b}, \theta)\, \psi_{(a,\boldsymbol{b},\theta)}(\boldsymbol{x})\, \frac{\mathrm{d}a}{a^3}\, \mathrm{d}\theta\, \mathrm{d}\boldsymbol{b}$$

with

$$c_\psi = \iint_{\mathbb{R}^2} \frac{|\widehat{\psi}(\boldsymbol{k})|^2}{\|\boldsymbol{k}\|^2}\, \mathrm{d}\boldsymbol{k}$$

The energy conservation writes:

$$\iint_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = \frac{1}{c_\psi} \int_0^{+\infty} \int_0^{2\pi} \iint_{\mathbb{R}^2} |Wf(a, \boldsymbol{b}, \theta)|^2\, \frac{\mathrm{d}a}{a^3}\, \mathrm{d}\theta\, \mathrm{d}\boldsymbol{b}$$

Inversion formula with a different wavelet

Let $f(\boldsymbol{x}) \in L^2(\mathbb{R}^2)$.

Wavelet decomposition of $f(\boldsymbol{x})$ with an *analysing wavelet* g:

$$W_g f(a, \boldsymbol{b}, \theta) = \iint_{\mathbb{R}^2} f(\boldsymbol{x}) \frac{1}{a} \bar{g} \left(\mathbf{R}_{-\theta} \left(\frac{\boldsymbol{x} - \boldsymbol{b}}{a} \right) \right) \mathrm{d}\boldsymbol{x}$$

Synthesis with a *reconstruction wavelet* h:

$$f(\boldsymbol{x}) = \frac{1}{c_{gh}} \int_0^{+\infty} \int_0^{2\pi} \iint_{\mathbb{R}^2} W_g f(a, \boldsymbol{b}, \theta) \frac{1}{a} h \left(\mathbf{R}_{-\theta} \left(\frac{\boldsymbol{x} - \boldsymbol{b}}{a} \right) \right) \frac{\mathrm{d}a}{a^3} \mathrm{d}\theta \, \mathrm{d}\boldsymbol{b}$$

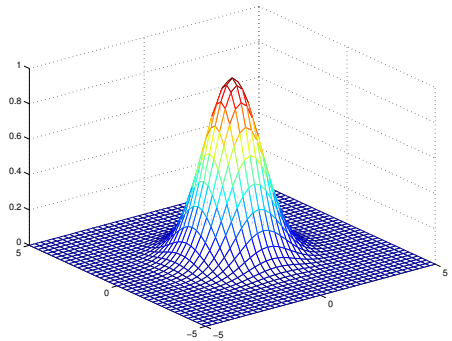
Cross admissibility condition on functions $g, h \in L^2(\mathbb{R}^2)$:

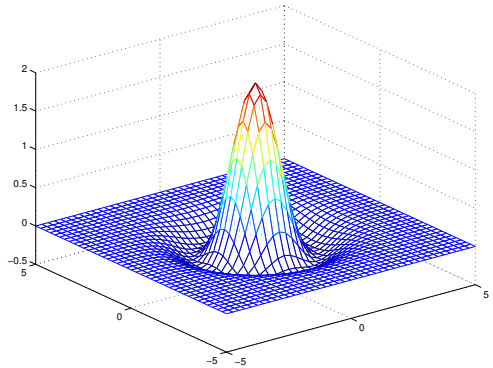
$$c_{gh} = \iint_{\mathbb{R}^2} \frac{\bar{\widehat{g}}(\boldsymbol{k}) \widehat{h}(\boldsymbol{k})}{\|\boldsymbol{k}\|^2} \mathrm{d}\boldsymbol{k} < +\infty$$

Classical examples

Wavelet constructed from the Gaussian $G(\boldsymbol{x}) = \mathrm{e}^{-\pi\|\boldsymbol{x}\|^2}$

- Wavelet transform with an **isotropic wavelet**
  - $g(\boldsymbol{x}) = h(\boldsymbol{x}) = \Delta G(\boldsymbol{x})$ *(Mexican hat)*
  - $g(\boldsymbol{x}) = G(\boldsymbol{x})$ (g is not a *wavelet*) and $h(\boldsymbol{x}) = \Delta G(\boldsymbol{x})$





- Wavelet transform with a **vector wavelet** $g(\boldsymbol{x}) = \nabla G(\boldsymbol{x})$ *(Canny multi-scale detector)*

Isotropic Wavelet Transform

When the wavelet is real, isotropic (i.e rotation invariant $\psi(\boldsymbol{x}) = h(\|\boldsymbol{x}\|)$), the wavelet transform of f comes down:

$$Wf(a, \boldsymbol{b}) = \frac{1}{a} \iint_{\mathbb{R}^2} f(\boldsymbol{x})\, \psi\left(\frac{\boldsymbol{x} - \boldsymbol{b}}{a}\right) \mathrm{d}\boldsymbol{x}$$

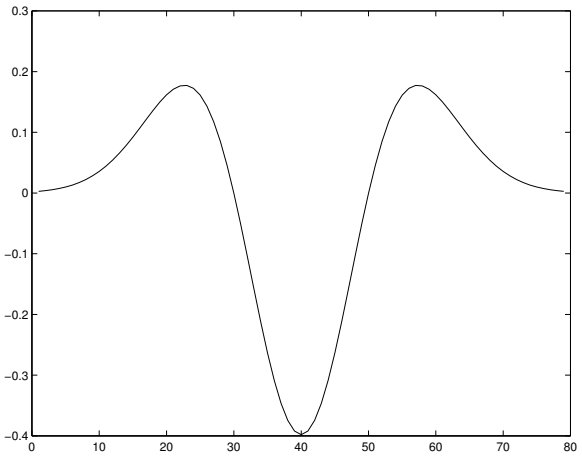
$\Rightarrow$ the integral on θ disappears

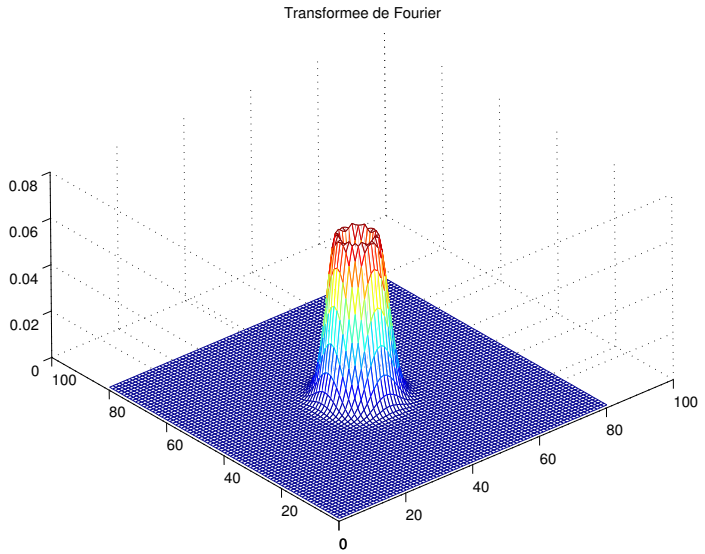
From Parseval equality, it writes:

$$Wf(a, \boldsymbol{b}) = a \iint_{\mathbb{R}^2} \widehat{f}(\boldsymbol{k})\, \overline{\widehat{\psi}(a\boldsymbol{k})} \mathrm{e}^{2i\pi \boldsymbol{k}\cdot\boldsymbol{b}}\, \mathrm{d}\boldsymbol{k}$$

$\Rightarrow$ the wavelet transform acts as a filter on the Fourier transform of f around the frequency $\frac{\boldsymbol{k}_0}{a}$.

Isotropic Wavelet Transform





If ψ is admissible, one has the energy conservation:

$$\iint_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = \frac{1}{c_\psi} \int_0^{+\infty} \iint_{\mathbb{R}^2} |Wf(a, \boldsymbol{b})|^2 \frac{\mathrm{d}a}{a^3} \, \mathrm{d}\boldsymbol{b}$$

and the synthesis formula:

$$f(\boldsymbol{x}) = \frac{1}{c_\psi} \int_0^{+\infty} \iint_{\mathbb{R}^2} Wf(a, \boldsymbol{b}) \psi_{a,\boldsymbol{b}}(\boldsymbol{x}) \frac{\mathrm{d}a}{a^3} \, \mathrm{d}\boldsymbol{b}$$

The Canny multiscale detector for image processing

Let Θ be a smoothing kernel such that:

- $\iint_{\mathbb{R}^2} \Theta = 1$
- $\Theta \geq 0$
- Θ isotropic or $\Theta(x, y) = \Theta_1(x)\Theta_2(y)$

Example (Gaussian)

$\Theta(\boldsymbol{x}) = G(\boldsymbol{x}) = \mathrm{e}^{-\pi\|\boldsymbol{x}\|^2}$ a smoothing kernel isotropic and tensorial

Directional Wavelets

$$\boldsymbol{\Psi}(x) = \nabla\Theta(x) = (\psi^1, \psi^2)$$

$$\psi^1 = -\frac{\partial\Theta}{\partial x_1} \quad \text{and} \quad \psi^2 = -\frac{\partial\Theta}{\partial x_2}$$

Wavelets in the direction $\varphi \Rightarrow \psi^\varphi = \cos\varphi\frac{\partial G}{\partial x_1} + \sin\varphi\frac{\partial G}{\partial x_2} = \vec{\varphi}\cdot\nabla G$

The Canny multiscale detector for image processing

Decomposition: computation of the vector wavelet transform

$$\boldsymbol{W} f(a, \boldsymbol{b}) = (W^1 f(a, \boldsymbol{b}), W^2 f(a, \boldsymbol{b}))$$

- $W^1 f(a, \boldsymbol{b}) = \iint_{\mathbb{R}^2} f(\boldsymbol{x}) \frac{1}{a} \psi^1 \left(\frac{\boldsymbol{x} - \boldsymbol{b}}{a} \right) \mathrm{d}\boldsymbol{x}$ $\rightarrow$ vertical singularities
- $W^2 f(a, \boldsymbol{b}) = \iint_{\mathbb{R}^2} f(\boldsymbol{x}) \frac{1}{a} \psi^2 \left(\frac{\boldsymbol{x} - \boldsymbol{b}}{a} \right) \mathrm{d}\boldsymbol{x}$ $\rightarrow$ horizontal singularities

Interpretation:

$$\boldsymbol{W} f(a, \boldsymbol{b}) = a \nabla \left(f * \frac{1}{a} \check{\Theta} \left(\frac{\boldsymbol{x}}{a} \right) \right) (\boldsymbol{b})$$

$\boldsymbol{W} f$ represents the gradient of the image, smoothed by Θ at scale a

Proof: Let define $L(\boldsymbol{x}) = -\frac{\boldsymbol{x}}{a}$, $\check{\Theta}_a = \frac{1}{a}\Theta \circ L$, $\check{\psi}_a^k(\boldsymbol{x}) = \frac{1}{a}\psi^k\left(-\frac{\boldsymbol{x}}{a}\right)$.

$$W^k f(a, \boldsymbol{b}) = \iint_{\mathbb{R}^2} f(\boldsymbol{x}) \frac{1}{a}\psi^k\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right) \mathrm{d}\boldsymbol{x} = (f * \check{\psi}_a^k)(\boldsymbol{b})$$

By the chain rule $\frac{\partial}{\partial x_k}(\Theta \circ L)(\boldsymbol{x}) = \frac{\partial \Theta}{\partial x_k}(L(\boldsymbol{x}))\frac{\partial L}{\partial x_k}(\boldsymbol{x})$ hence

$$\check{\psi}_a^k(\boldsymbol{x}) = -\frac{1}{a}\frac{\partial \Theta}{\partial x_k}\left(-\frac{\boldsymbol{x}}{a}\right) = a\frac{\partial}{\partial x_k}\left(\frac{1}{a}\Theta \circ L\right)(\boldsymbol{x}) = a\frac{\partial \check{\Theta}_a}{\partial x_k}$$

$$\begin{aligned}
(f * \check{\psi}_a^k)(\boldsymbol{b}) &= a\iint f(\boldsymbol{x})\frac{\partial \check{\Theta}_a}{\partial x_k}(\boldsymbol{b}-\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&= \frac{\partial}{\partial b_k}\iint f(\boldsymbol{x})\check{\Theta}_a(\boldsymbol{b}-\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&= \frac{\partial}{\partial b_k}(f * \check{\Theta}_a)(\boldsymbol{b})
\end{aligned}$$

$$\begin{pmatrix} W^1 f(a, \boldsymbol{b}) \\ W^2 f(a, \boldsymbol{b}) \end{pmatrix} = a\begin{pmatrix} \frac{\partial}{\partial b_1}(f * \check{\Theta}_a)(\boldsymbol{b}) \\ \frac{\partial}{\partial b_2}(f * \check{\Theta}_a)(\boldsymbol{b}) \end{pmatrix} = a\nabla(f * \check{\Theta}_a)(\boldsymbol{b}) \quad \square$$

Multiscale detector and directional wavelet transform

If Θ is isotropic, one has $\frac{\partial \Theta}{\partial x_1}(\mathbf{R}^{-\theta}\boldsymbol{x}) = \cos\theta\, \frac{\partial \Theta}{\partial x_1}(\boldsymbol{x}) + \sin\theta\, \frac{\partial \Theta}{\partial x_2}(\boldsymbol{x})$
Then,

$$W_{\psi^1} f(a, \boldsymbol{b}, \varphi) = \vec{\varphi} \cdot \boldsymbol{W} f(a, \boldsymbol{b}) \rightarrow \text{singularities in the direction } \vec{\varphi}^{\perp}$$

where $W_{\psi^1} f$ is the directional wavelet transform of f with ψ^1 as analyzing wavelet.

One can write:

$$\begin{pmatrix} W_{\psi^1} f(a, \boldsymbol{b}, \varphi) \\ W_{\psi^2} f(a, \boldsymbol{b}, \varphi) \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} W^1 f(a, \boldsymbol{b}) \\ W^2 f(a, \boldsymbol{b}) \end{pmatrix}$$

In vector formulation:

$$\boldsymbol{W}_{\nabla\Theta} = \mathbf{R}_{-\varphi} \boldsymbol{W}$$

$\rightsquigarrow$ will provide a reconstruction formula for the multiscale detector!

Inversion of the multiscale detector (Le Cadet, PhD 2004)

$$f(\boldsymbol{x}) = \frac{\pi}{C_{\psi^1}} \int_{a>0} \frac{\mathrm{d}a}{a^3} \iint_{\mathbb{R}^2} \boldsymbol{W} f(a, \boldsymbol{b}) \cdot \psi_{a,\boldsymbol{b}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{b}$$

with $C_{\psi^1} = \pi^2$ for $\Theta = G$.

Proof: The reconstruction formula of the directional wavelet transform with wavelet ψ^1 gives:

$$f(\boldsymbol{x}) = \frac{1}{C_{\psi^1}} \int_0^{2\pi} \int_0^{+\infty} \iint_{\mathbb{R}^2} W_{\psi^1} f(a, \boldsymbol{b}, \theta) \frac{1}{a} \psi^1 \left(\mathbf{R}^{-\theta} \left(\frac{\boldsymbol{x} - \boldsymbol{b}}{a} \right) \right) \mathrm{d}\theta \frac{\mathrm{d}a}{a^3} \, \mathrm{d}\boldsymbol{b}$$

Replacing $W_{\psi^1} f(a, \boldsymbol{b}, \theta)$ by its definition:

$$\begin{aligned} f(\boldsymbol{x}) &= \frac{1}{C_{\psi^1}} \int_0^{2\pi} \int_0^{+\infty} \iint_{\mathbb{R}^2} \left[\cos\theta \, W^1 f(a, \boldsymbol{b}) + \sin\theta \, W^2 f(a, \boldsymbol{b}) \right] \\ &\quad \frac{1}{a} \frac{\partial \Theta}{\partial x_1} \left(\mathbf{R}^{-\theta} \left(\frac{\boldsymbol{x} - \boldsymbol{b}}{a} \right) \right) \mathrm{d}\theta \frac{\mathrm{d}a}{a^3} \, \mathrm{d}\boldsymbol{b} \end{aligned}$$

If Θ is isotropic, one has:

$$\frac{\partial \Theta}{\partial x_1}(\mathbf{R}^{-\theta}\boldsymbol{x}) = \cos\theta\,\frac{\partial \Theta}{\partial x_1}(\boldsymbol{x}) + \sin\theta\,\frac{\partial \Theta}{\partial x_2}(\boldsymbol{x})$$

then

$$\begin{aligned}
f(\boldsymbol{x})C_{\psi^1} &= \int_{a>0}\frac{\mathrm{d}a}{a^3}\iint_{\mathbb{R}^2}\mathrm{d}\boldsymbol{b}\int_0^{2\pi}\mathrm{d}\theta\cos^2\theta\,W^1 f(a,\boldsymbol{b})\,\frac{1}{a}\frac{\partial\Theta}{\partial x_1}\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)\\
&+ \int_{a>0}\frac{\mathrm{d}a}{a^3}\iint_{\mathbb{R}^2}\mathrm{d}\boldsymbol{b}\int_0^{2\pi}\mathrm{d}\theta\sin^2\theta\,W^2 f(a,\boldsymbol{b})\,\frac{1}{a}\frac{\partial\Theta}{\partial x_2}\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)\\
&+ \int_{a>0}\frac{\mathrm{d}a}{a^3}\iint_{\mathbb{R}^2}\mathrm{d}\boldsymbol{b}\int_0^{2\pi}\mathrm{d}\theta\cos\theta\sin\theta\,W^1 f(a,\boldsymbol{b})\,\frac{1}{a}\frac{\partial\Theta}{\partial x_2}\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)\\
&+ \int_{a>0}\frac{\mathrm{d}a}{a^3}\iint_{\mathbb{R}^2}\mathrm{d}\boldsymbol{b}\int_0^{2\pi}\mathrm{d}\theta\cos\theta\sin\theta\;W^2 f(a,\boldsymbol{b})\,\frac{1}{a}\frac{\partial\Theta}{\partial x_1}\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)
\end{aligned}$$

Since $\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta = \int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta = \pi$ and $\int_0^{2\pi}\cos\theta\sin\theta\,\mathrm{d}\theta = 0$ then

$$f(\boldsymbol{x}) = \frac{\pi}{C_{\psi^1}}\int_{a>0}\frac{\mathrm{d}a}{a^3}\iint_{\mathbb{R}^2}\mathrm{d}\boldsymbol{b}\left[W^1 f(a,\boldsymbol{b})\psi^1_{a,\boldsymbol{b}}(\boldsymbol{x}) + W^2 f(a,\boldsymbol{b})\psi^2_{a,\boldsymbol{b}}(\boldsymbol{x})\right]\ \square$$

Energy conservation formula

The vector $\boldsymbol{W}f(a,\boldsymbol{b})$ should be represented in modulus-orientation:

$$\begin{array}{rcll} Mf(a,\boldsymbol{b}) & = & \|\boldsymbol{W}f(a,\boldsymbol{b})\| & \text{Modulus} \\ Af(a,\boldsymbol{b}) & = & \operatorname{Arg}\left(\boldsymbol{W}f(a,\boldsymbol{b})\right) & \text{Orientation} \end{array}$$

The energy conservation (with an isotropic kernel Θ) writes:

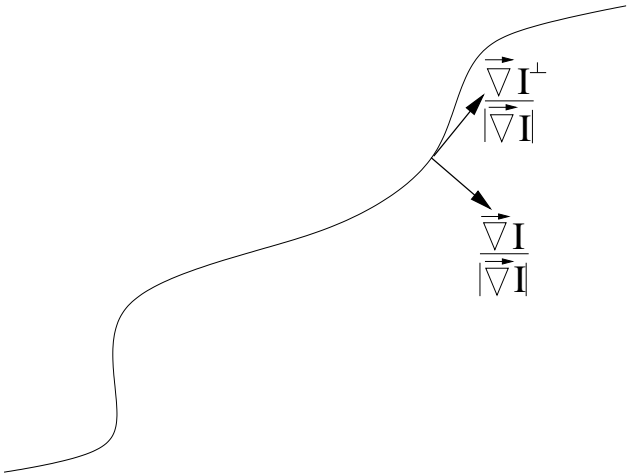
$$\iint_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = \frac{\pi}{C_\psi} \int_{a>0} \frac{\mathrm{d}a}{a^3} \iint_{\mathbb{R}^2} (Mf(a,\boldsymbol{b}))^2 \,\mathrm{d}\boldsymbol{b}$$

Example (Application: edge detection in 2D images)

1. Edge points at scale a are points where $\boldsymbol{b} \mapsto Mf(a,\boldsymbol{b})$ is locally maximum in the direction $Af(a,\boldsymbol{b})$.
2. Estimation of the maxima lines linking edge points through scales a. The *tops* of these maxima lines ($a \to 0$) finally constitute the edge points of the image.
3. Computation of the Lipschitz regularity at any edge point.

Application: edge detection in 2D images

Detection and classification of edges of a regular image, regular outside regular singularity lines.



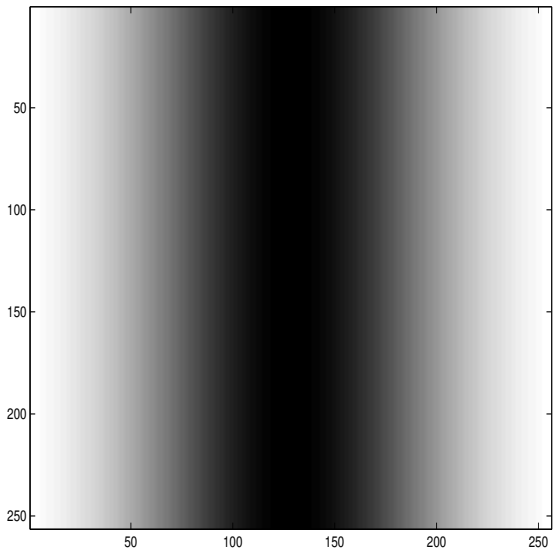
The edge is characterized by a singularity in the intensity, in the direction of the gradient $\vec{\nabla} I$*. Along the edge, i.e. in the orthogonal direction of the gradient, the regularity is maximal*

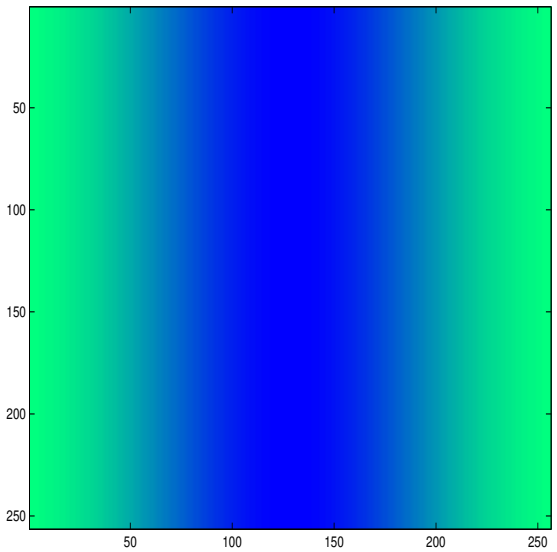
Remark: In pratice one will consider $\nabla(I * \frac{1}{a^2} G(\frac{x}{a}))$, which correspond to wavelet coefficients of I with a wavelet *Gaussian gradient*.

Application: edge detection in 2D images

Edge Model (Canny 86)

A point (x_0, y_0) of an image is an edge point if at this point the gradient modulus of the intensity, smoothed by a kernel θ_a, $|\nabla(I * \theta_a)|$, is locally maximum in the direction of the gradient $\nabla(I * \theta_a)$.





Variation of the intensity of a Gaussian distribution; where are the edges?

Application: edge detection in 2D images

New Edge Model (Mallat-Zhong, Mallat-Hwang 92, Le Cadet 2004)

f image smoothed by a kernel θ_a of scale a with $0 < a < a_{max}$: $g_a = f * \theta_a$. If there exists a connected curve through scales, along which all points are local maxima in the gradient direction ∇g_a, the limit (x_0, y_0) of this curve at small scales is an edge point.

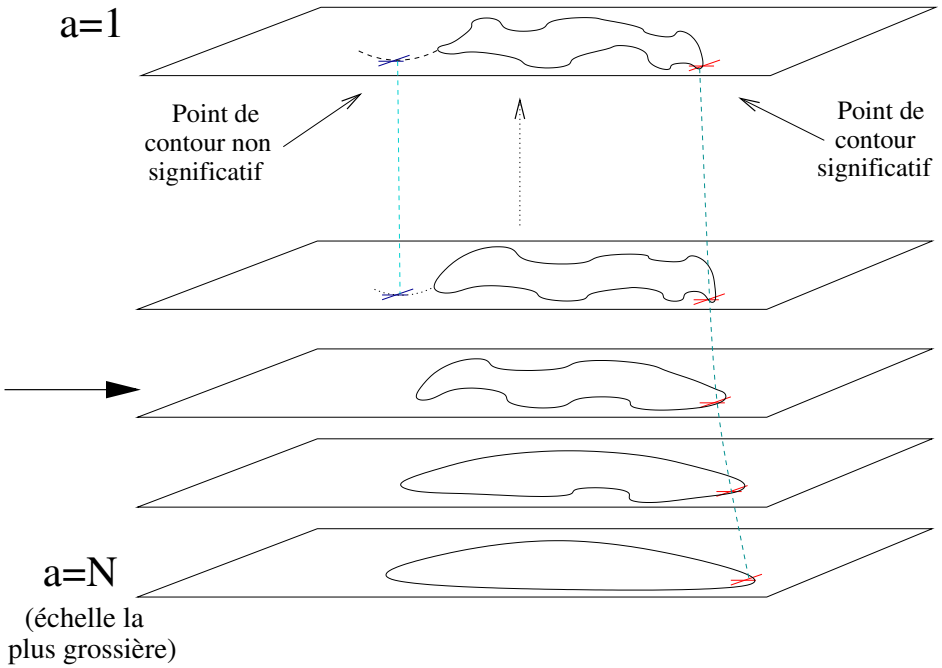
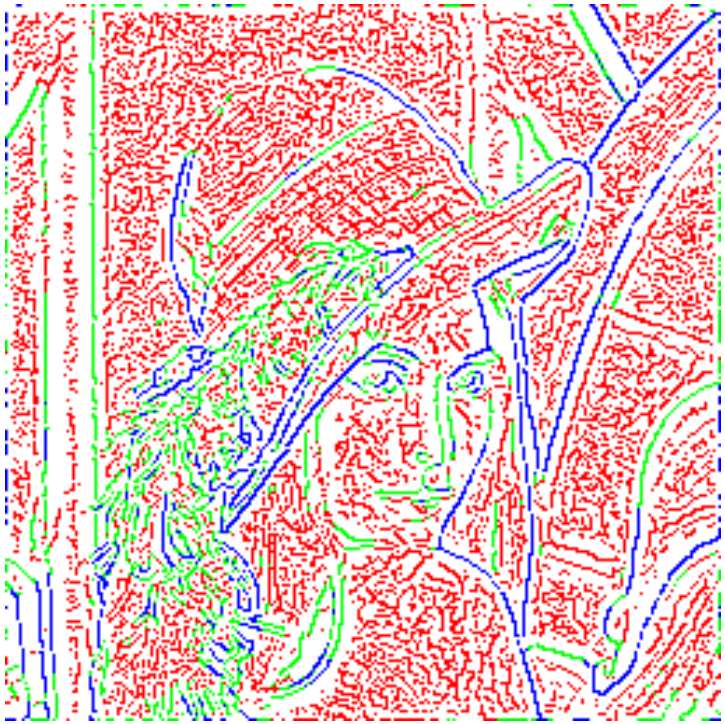


Figure: 2D maxima lines

Application: modulus of the wavelet transform local max.

Fine scale

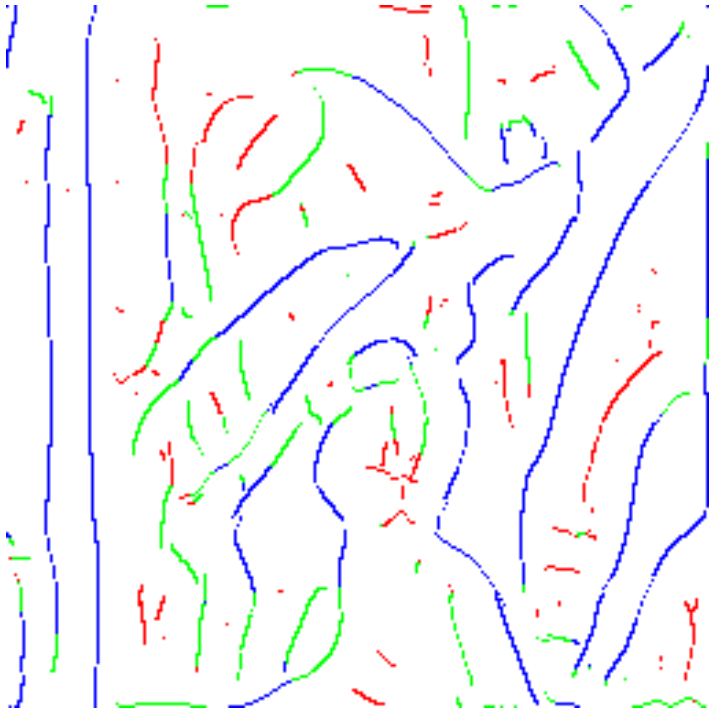


Application: modulus of the wavelet transform local max.

Intermediate scale

Application: modulus of the wavelet transform local max.

Large scale



Application: modulus of the wavelet transform local max.

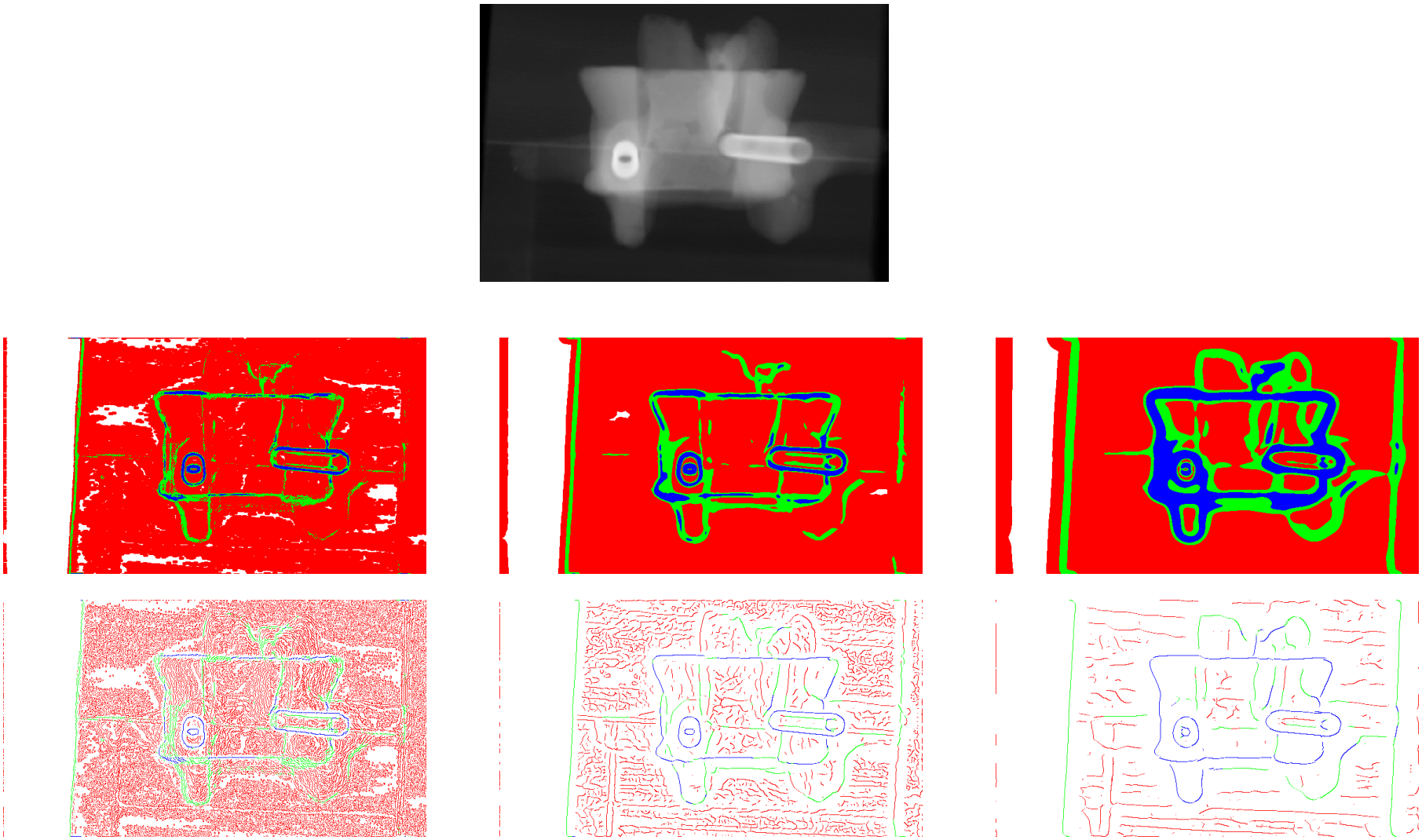
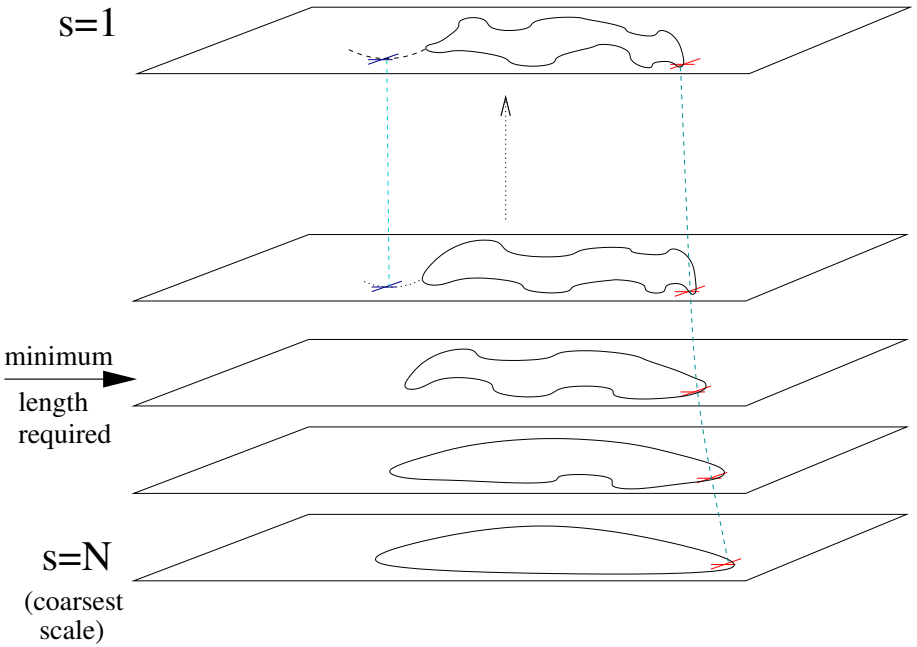
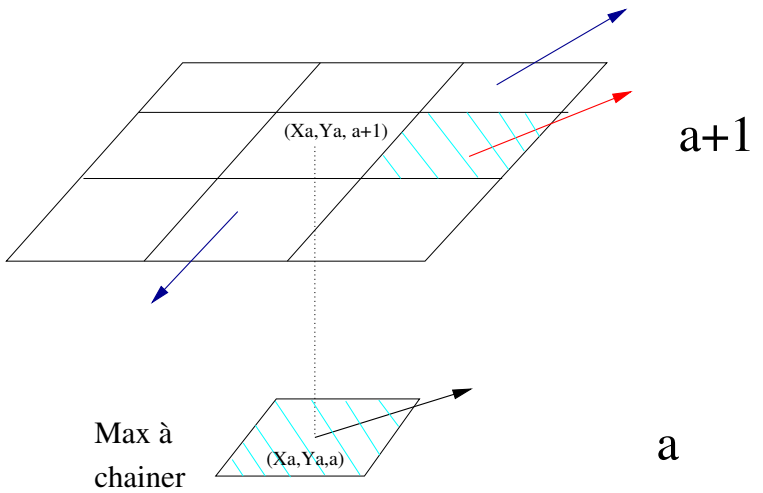


Figure: Edge points (top), wavelet coefficients maps at fixed scale (bottom) of a X-Ray image

Practice of the maxima line construction in 2D

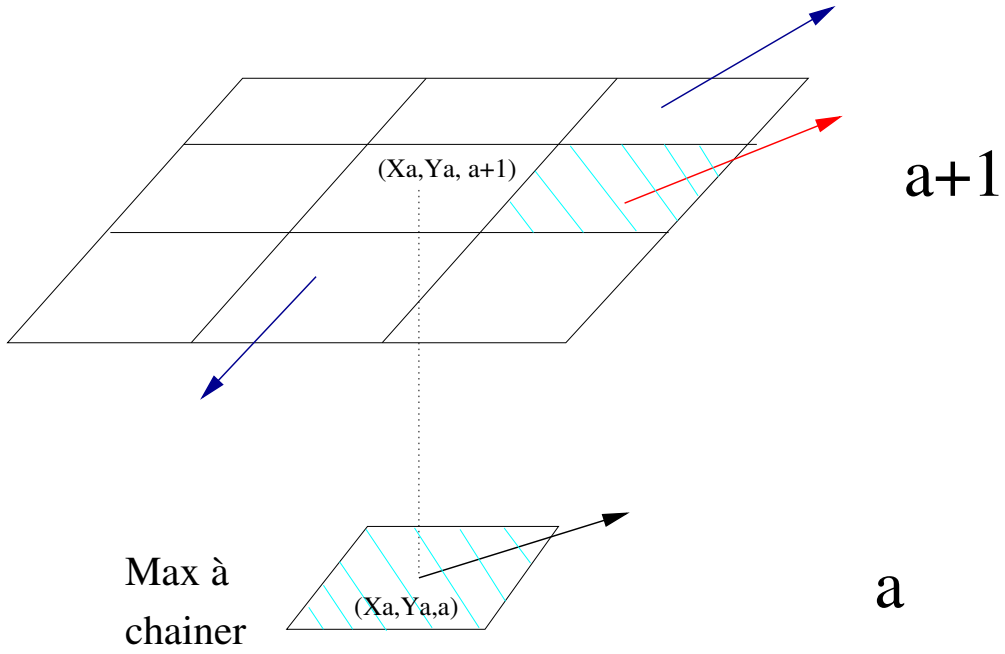
1. Map of modulus maxima (in the gradient direction) at each scale.
2. Two modulus maxima between two successive scales are linked if they are neighbors in the gradient direction.





Practice of the maxima line construction in 2D

1. Let $Mf(x_0, y_0, a_{dep})$ be a modulus maximum at scale a_{dep}.
2. One consider, the 9 modulus $Mf(x_0(\pm 1), y_0(\pm 1), a_{dep+1})$.
3. One links with the maximum modulus that has the angle $Af(x_1, y_1, a_{dep+1})$ closest to $Af(x_0, y_0, a_{dep})$.



The dyadic wavelet transform

The scale varies along the dyadic sequence $\{2^j\}_{j\in\mathbb{Z}}$. Let $1 \leq k \leq 2$

$$\psi^k(\boldsymbol{x}) = -\frac{\partial\theta}{\partial x_k}, \quad \psi^k_{2^j}(\boldsymbol{x}) = \frac{1}{2^j}\psi^k\left(\frac{\boldsymbol{x}}{2^j}\right), \quad \check{\psi}^k_{2^j}(\boldsymbol{x}) = \psi^k_{2^j}(-\boldsymbol{x})$$

The dyadic wavelet transform at $\boldsymbol{b} = (b_1, b_2)$ is:

$$W^k(2^j, \boldsymbol{b}) = \langle f, \psi^k_{2^j}(\cdot - \boldsymbol{b})\rangle = f * \check{\psi}^k_{2^j}(\boldsymbol{b})$$

Let $\theta_{2^j}(\boldsymbol{x}) = 2^{-j}\theta(2^{-j}\boldsymbol{x})$ and $\check{\theta}_{2^j}(\boldsymbol{x}) = \theta_{2^j}(-\boldsymbol{x})$. The wavelet transform components are proportional to the gradient of f smoothed by $\check{\theta}_{2^j}$:

$$\begin{pmatrix} W^1 f(2^j, \boldsymbol{b}) \\ W^2 f(2^j, \boldsymbol{b}) \end{pmatrix} = 2^j \begin{pmatrix} \frac{\partial}{\partial b_1}(f * \check{\theta}_{2^j})(\boldsymbol{b}) \\ \frac{\partial}{\partial b_2}(f * \check{\theta}_{2^j})(\boldsymbol{b}) \end{pmatrix} = 2^j \nabla(f * \check{\theta}_{2^j})(\boldsymbol{b})$$

The dyadic wavelet transform

- The modulus of this gradient vector is proportional to the wavelet transform modulus:

$$Mf(2^j, \boldsymbol{b}) = \sqrt{|W^1 f(2^j, \boldsymbol{b})|^2 + |W^2 f(2^j, \boldsymbol{b})|^2}$$

- The angle $Af(2^j, \boldsymbol{b})$ of the wavelet transform vector:

$$Af(2^j, \boldsymbol{b}) = \begin{cases} \alpha(\boldsymbol{b}) & \text{if } W^1 f(2^j, \boldsymbol{b}) \geqslant 0 \\ \pi + \alpha(\boldsymbol{b}) & \text{if } W^2 f(2^j, \boldsymbol{b}) \geqslant 0 \end{cases},$$

$$\alpha(\boldsymbol{b}) = \tan^{-1}\left[\frac{W^2 f(2^j, \boldsymbol{b})}{W^1 f(2^j, \boldsymbol{b})}\right], \boldsymbol{n}_j(\boldsymbol{b}) = (\cos Af(2^j, \boldsymbol{b}),\ \sin Af(2^j, \boldsymbol{b}))$$

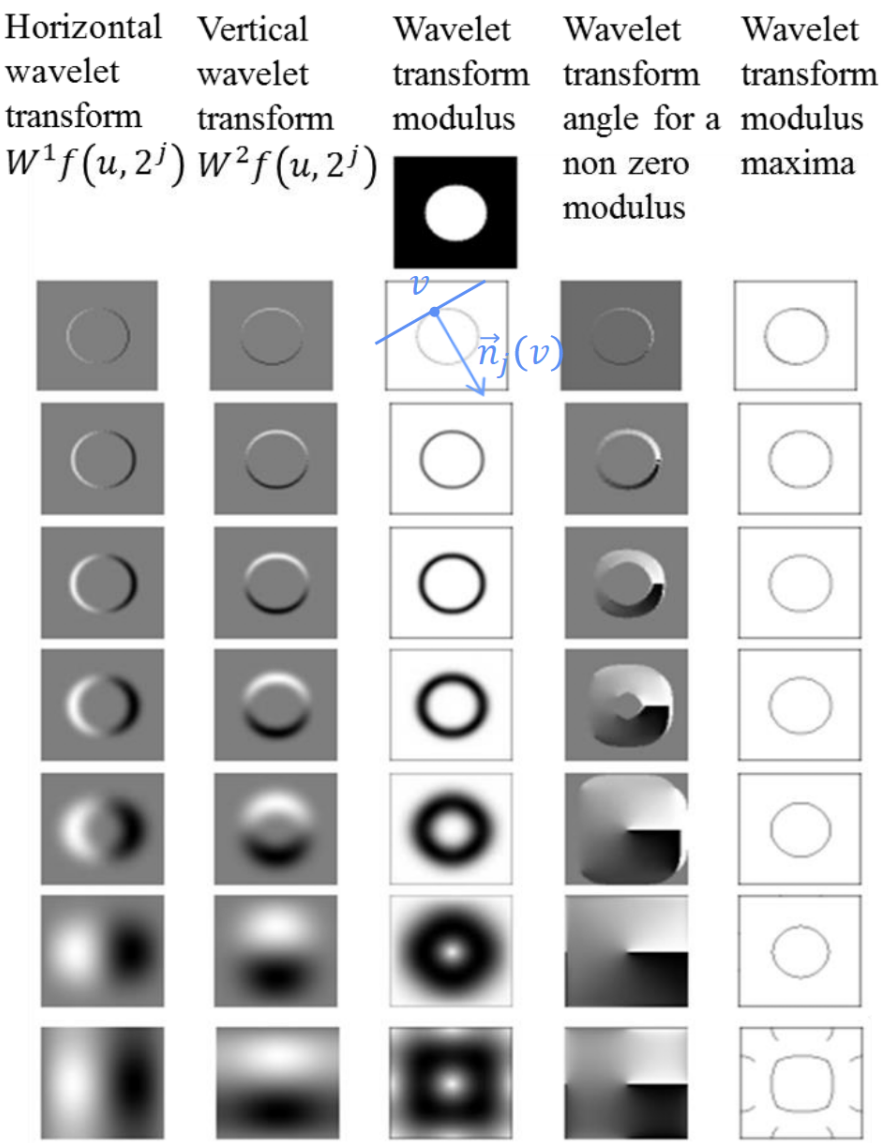
- An edge point $\boldsymbol{b}_0$ at the scale 2^j: $Mf(2^j, \boldsymbol{b})$ is locally maximum at $\boldsymbol{b} = \boldsymbol{b}_0$ when $\boldsymbol{b} = \boldsymbol{b}_0 + \lambda \boldsymbol{n}_j(\boldsymbol{b}_0)$ and $|\lambda|$ small enough.
- The level sets of $g(\boldsymbol{x})$ are the curves $\boldsymbol{x}(s)$ where $g(\boldsymbol{x}(s))$ is constant. If $\boldsymbol{\tau} \perp \boldsymbol{x}(s)$ then

$$\frac{\partial \boldsymbol{x}(s)}{\partial s} = \nabla g \cdot \boldsymbol{\tau} = 0$$

The dyadic wavelet transform

- The level set property applied to $g = f * \check{\theta}_{2^j}$ proves that a maximum point $\boldsymbol{b}_0$ the vector $\boldsymbol{n}_j(\boldsymbol{b}_0)$ of angle $Af(2^j, \boldsymbol{b}_0)$ is perpendicular to the level set of $f * \check{\theta}_{2^j}$ going through $\boldsymbol{b}_0$.

- If the intensity profile remains constant along an edge, then the inflection points (maxima points) are along a level set. The intensity profile of an edge may not be constant but its variations are often negligible over a neighborhood of size 2^j for a small scale 2^j. The tangent of the maxima curve is then nearly perpendicular to $\boldsymbol{n}_j(\boldsymbol{b}_0)$



Reconstruction of edge curves

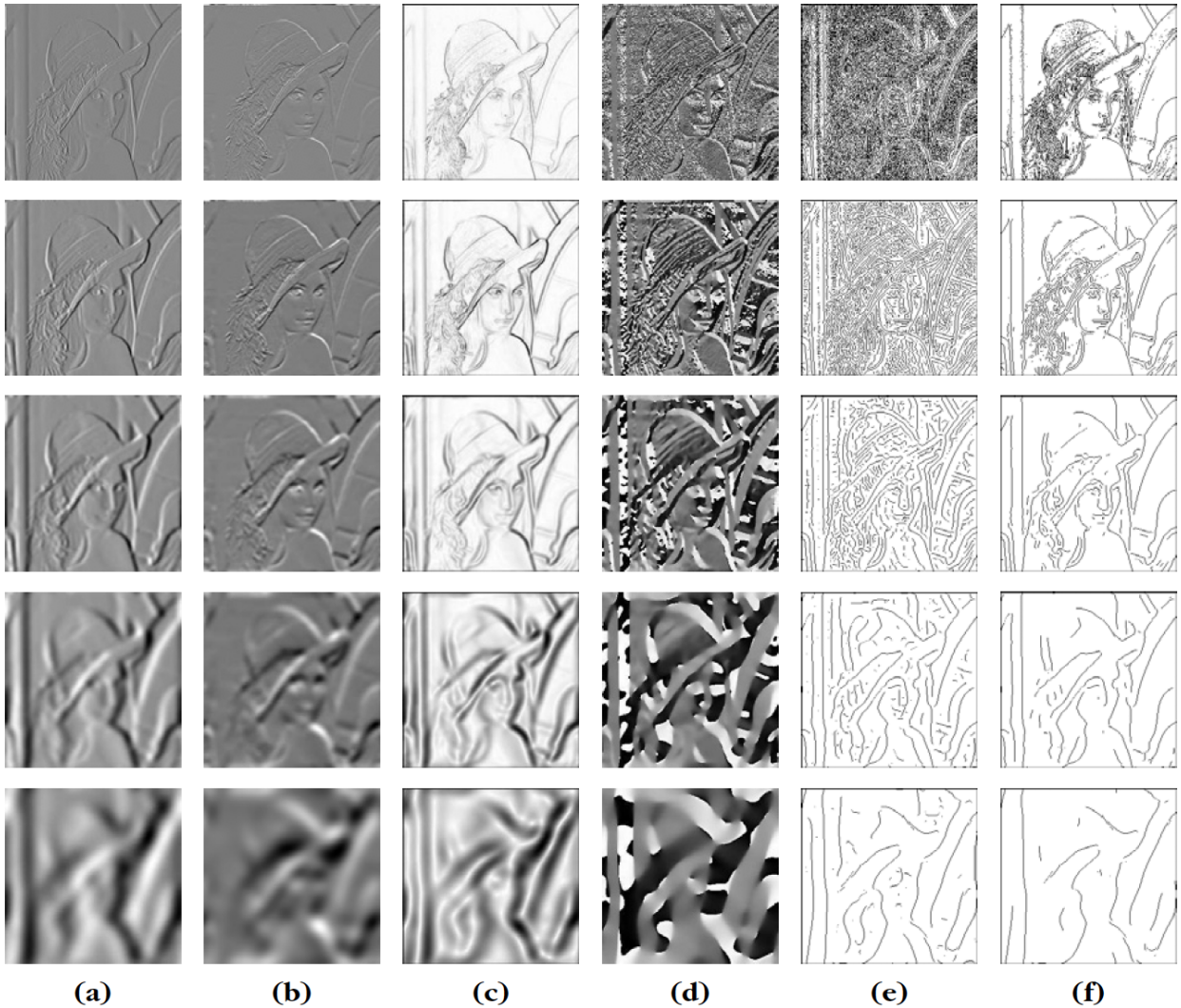


FIGURE 6.11

Multiscale edges of the Lena image shown in Figure 6.12. **(a)** $\{W^1 f(u, 2^j)\}_{-7 \leqslant j \leqslant -3}$. **(b)** $\{W^2 f(u, 2^j)\}_{-7 \leqslant j \leqslant -3}$. **(c)** $\{Mf(u, 2^j)\}_{-7 \leqslant j \leqslant -3}$. **(d)** $\{Af(u, 2^j)\}_{-7 \leqslant j \leqslant -3}$. **(e)** Modulus maxima support. **(f)** Support of maxima with modulus values above a threshold.

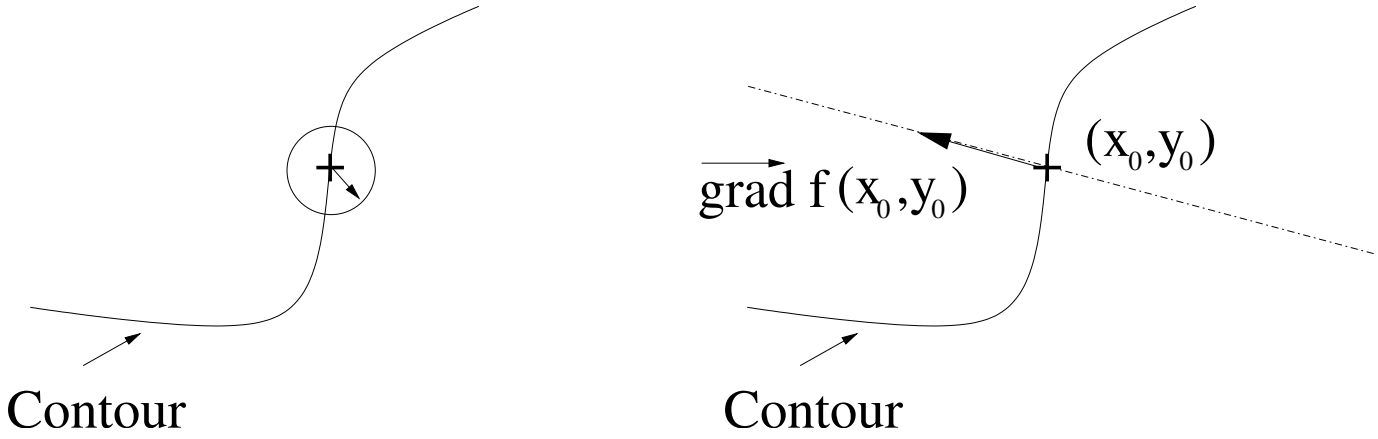
Credits: Mallat

Application: characterization of the singularities

Regularity of edge curves

Let $0 \leq \alpha < 1$. $f(x, y)$ Lipschitz-α at (x_0, y_0) if $\exists A$ s.t $\forall \boldsymbol{h} = (h_1, h_2)$,

$$|f(x_0 + h_1, y_0 + h_2) - f(x_0, y_0)| \leq A\|\boldsymbol{h}\|^{\alpha}$$



On a curve of discontinuity, the estimation of the regularity reduces to the one dimensional case. f is uniformly Lipschitz-α inside Ω iff

$$\forall (x, y) \in \Omega,\ \forall j, \quad |Mf(x, y, 2^j)| \leqslant A\, 2^{j(\alpha+1)}$$

Computation of the Lipschitz regularity

The Lipschitz regularity is evaluated at each edge point, by computing the slope of $\log Mf(x_c, y_c, a) = g(\log a)$

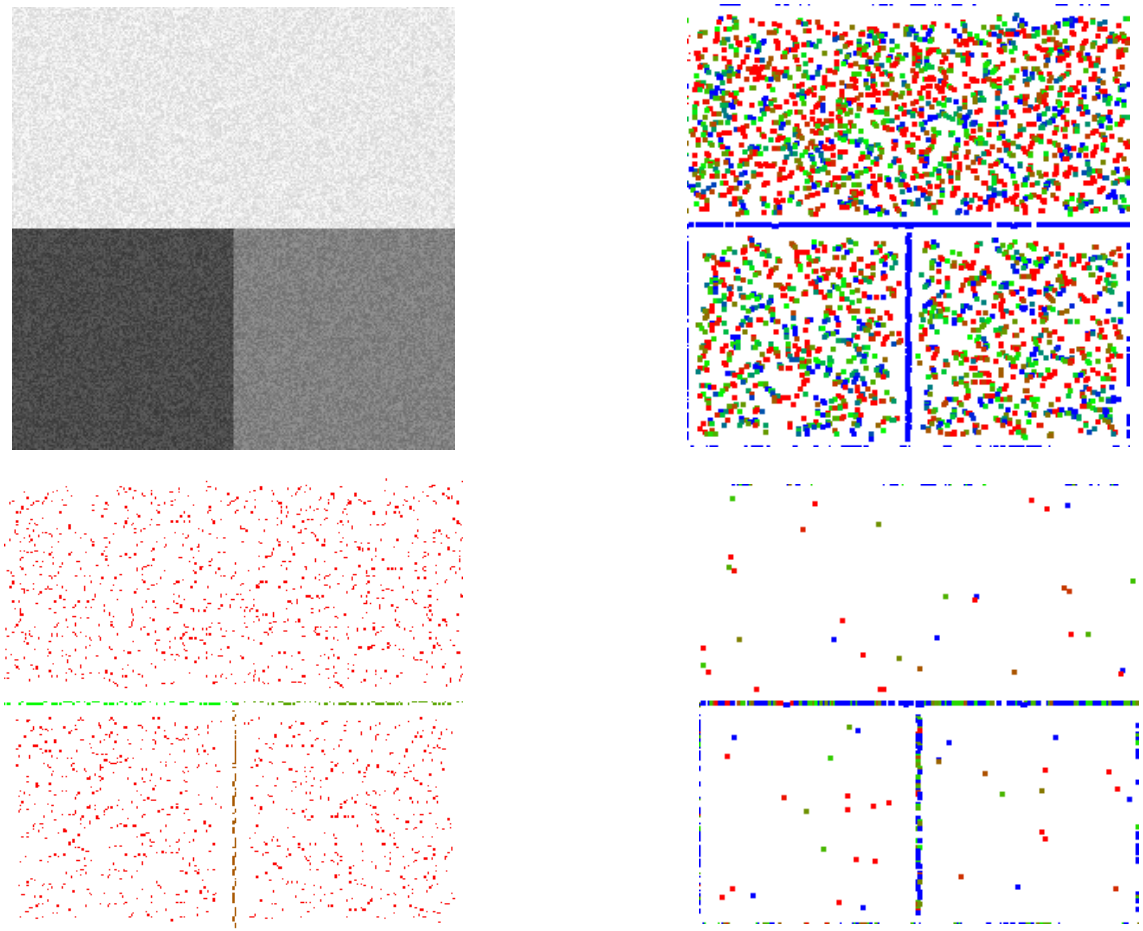


Figure: Three noisy domain: maps of modulus, Lipschitz regularity, denoising

Examples

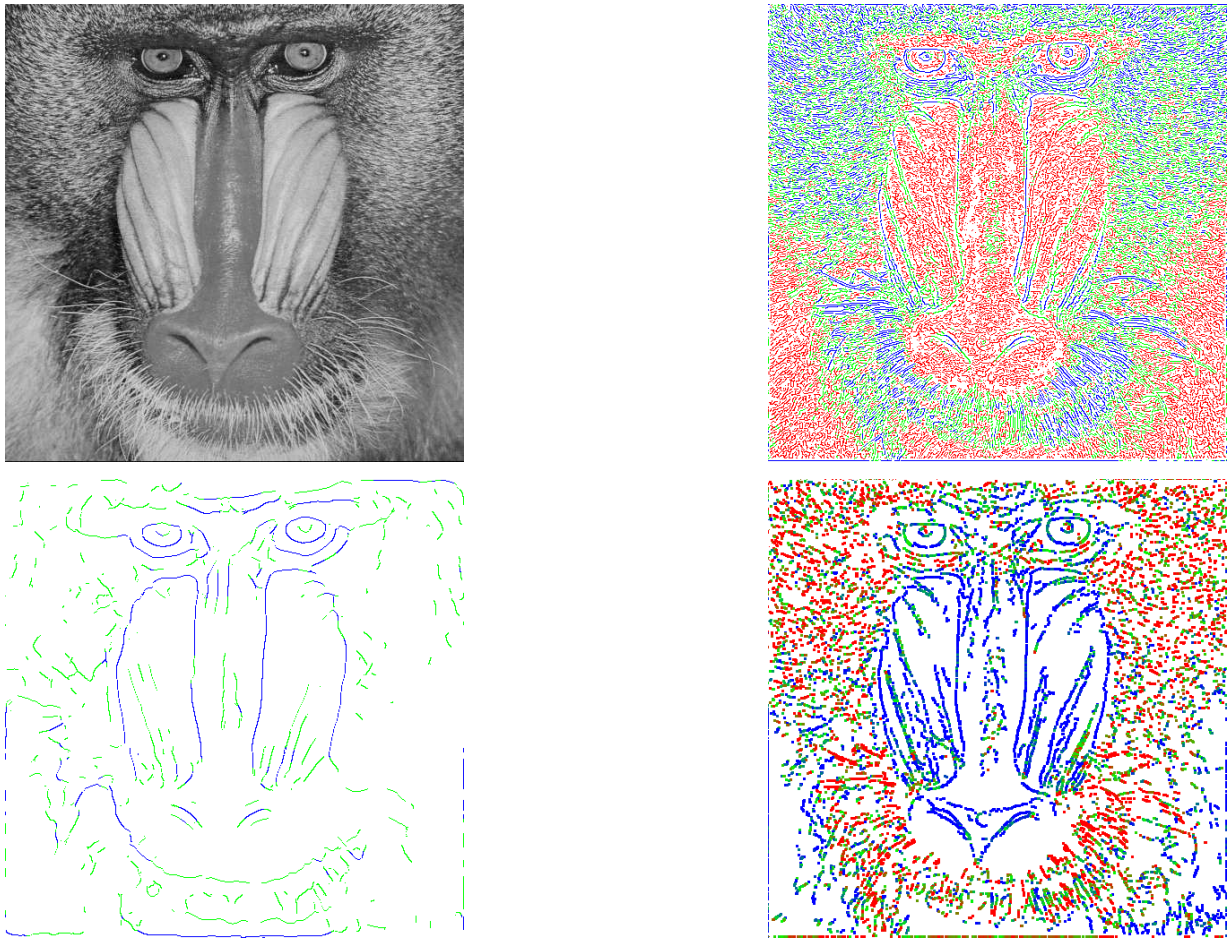


Figure: Mandrill original image (top left), large scale edge points (bottom left) and fine scale edge points (top right) and local regularities computed on maxima lines (bottom right)

Examples

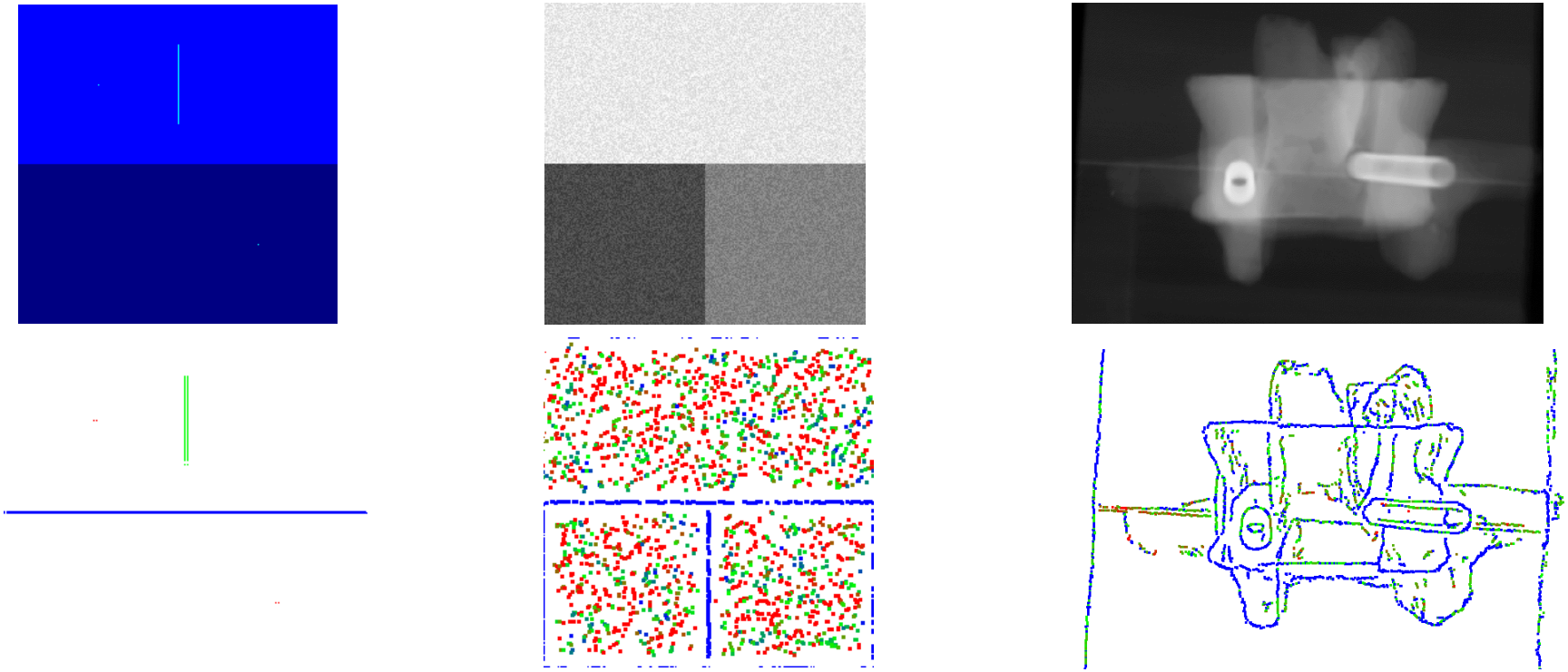


Figure: Top: original images; Bottom: edge points (the colors represent the regularity parameter)

Examples

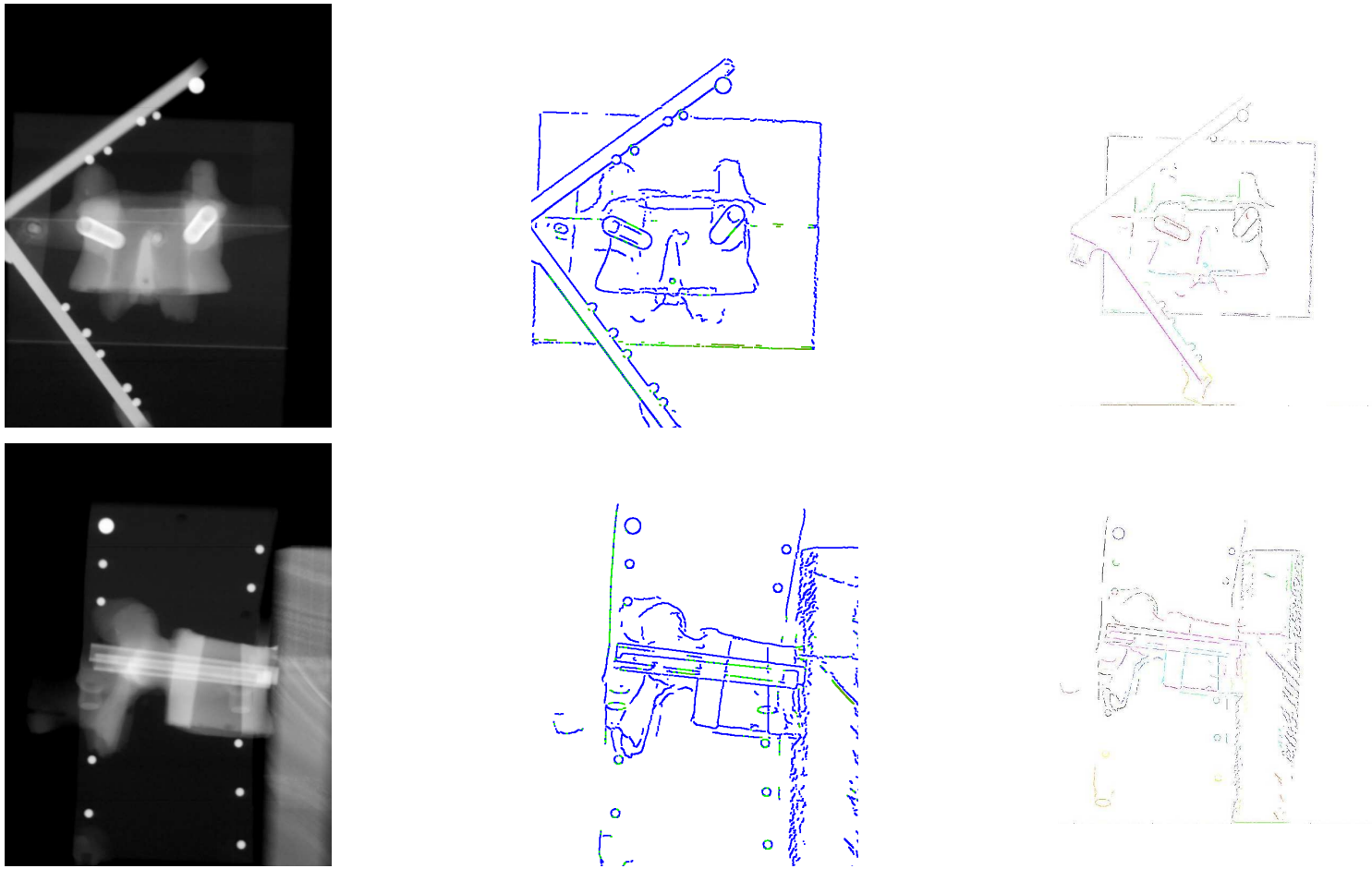


Figure: Edge detection on two X-rays of vertebra

Reconstruction from edges

Image approximations can be computed by projecting the image on the space generated by wavelets on the modulus maxima support. Let Λ be the set of all modulus maxima points $(2^j, \boldsymbol{b})$, $\boldsymbol{n}$ is the unit vector in the direction $Af(2^j, \boldsymbol{b})$ and

$$\psi^3_{2^j,\boldsymbol{b}}(\boldsymbol{x}) = 2^{2j} \frac{\partial^2 \theta_{2^j}(\boldsymbol{x} - \boldsymbol{b})}{\partial \boldsymbol{n}^2}$$

Since the wavelet modulus $Mf(2^j, \boldsymbol{b})$ has a local maximum at $\boldsymbol{b}$ in the direction of $\boldsymbol{n}$ then $\langle f, \psi^3_{2^j,\boldsymbol{b}} \rangle = 0$.

A modulus maxima approximation f_Λ can be computed as an **orthonormal projection** of f on the space generated by the family of maxima wavelets $\{\psi^k_{2^j,\boldsymbol{b}}\}_{(2^j,\boldsymbol{b})\in\Lambda, 1\leq k\leq 3}$:

$$f_\Lambda = \mathbf{L}^{-1}(\mathbf{L}f), \quad \mathbf{L}y = \sum_{(2^j,\boldsymbol{b})\in\Lambda} \sum_{k=1}^{2} \langle y, \psi^k_{2^j,\boldsymbol{b}} \rangle \psi^k_{2^j,\boldsymbol{b}}$$

Credits: Mallat (see chapter 5 on frames and especially section 5.1.3 on dual synthesis)

Reconstruction from edges

(a) (b)

(c)

FIGURE 6.12

(a) Original Lena image. **(b)** Image reconstructed from the wavelet maxima displayed in Figure 6.11(e) and larger-scale maxima. **(c)** Image reconstructed from the thresholded wavele maxima displayed in Figure 6.11(f) and larger-scale maxima.

Credits: Mallat

Denoising by multiscale edge thresholding

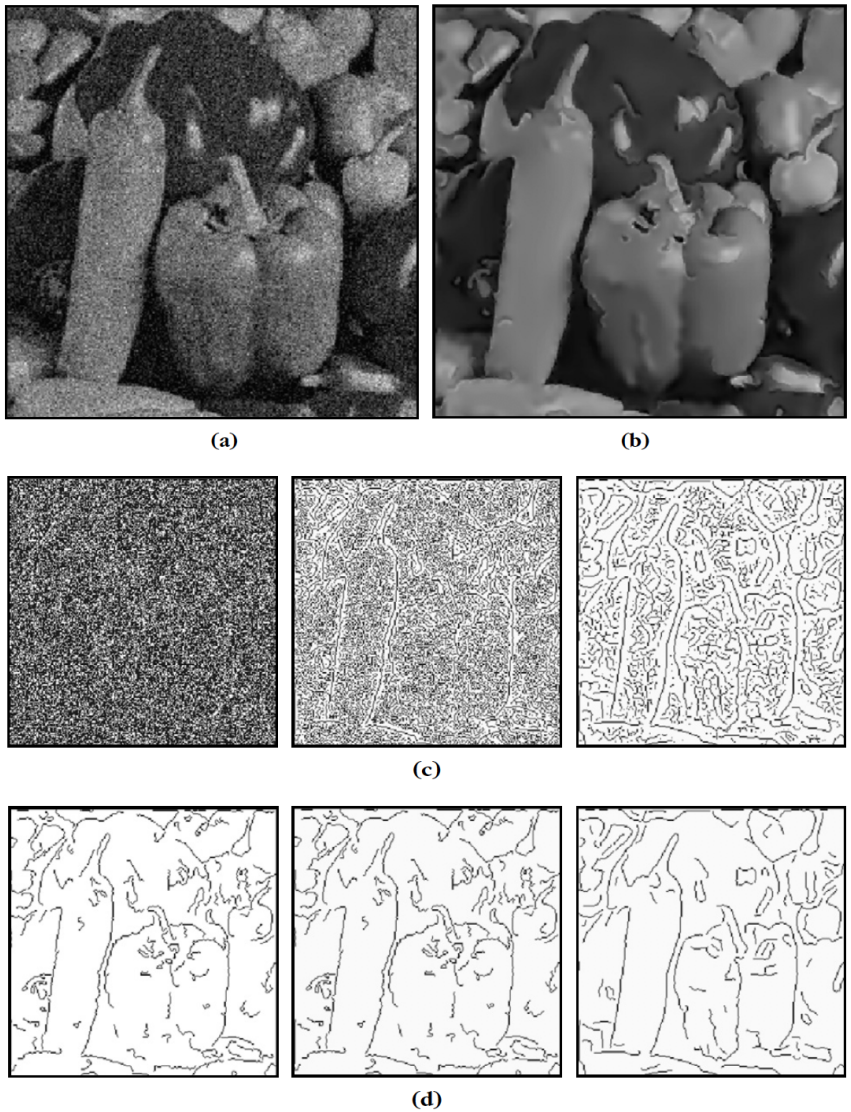


FIGURE 6.13

(a) Noisy peppers image. **(b)** Peppers image restored from the thresholding maxima chains shown in **(d)**. The images in row **(c)** show the wavelet maxima support of the noisy image—the scale increases from left to right, from 2^{-7} to 2^{-5}. The images in row (d) give the maxima support computed with a thresholding selection of the maxima chains.

Credits: Mallat