Wavelets and Applications

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M2 MSIAM & Ensimag 3A MMIS

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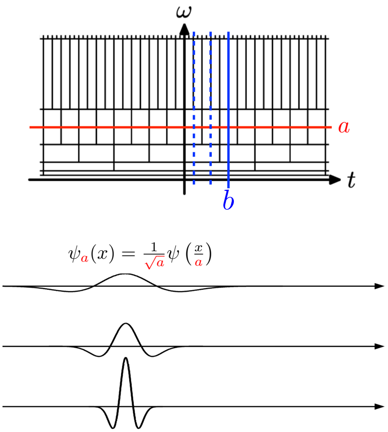
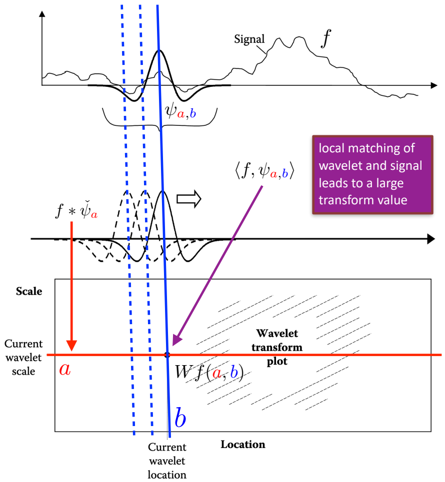
The 1D Discrete Wavelet Transform

From the CWT to the DWT





Scaling function

When $Wf(a, b)$ is known only for $a < a_0$, to recover f we need a complement of information that corresponds to $Wf(a, b)$ for $a > a_0$. This is obtained by introducing a scaling function ϕ that is an aggregation of wavelets at scales larger than 1:

$$|\widehat{\phi}(\omega)|^2 = \int_1^{+\infty} |\widehat{\psi}(a\,\omega)|^2 \frac{\mathrm{d}a}{a} = \int_\omega^{+\infty} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|} \mathrm{d}\xi$$

and the complex phase of $\widehat{\phi}(\omega)$ can be arbitrarily chosen. One can verify that $\|\phi\| = 1$, and from admissibility condition that $\lim_{\omega\to 0}|\widehat{\phi}(\omega)|^2 = C_\psi$. The scaling function therefore can be interpreted as the impulse response of a **low-pass filter**. Let us denote $\phi_a(x) = a^{-1/2}\phi(x/a)$ and $\check{\phi}_a(x) = \phi_a^*(-x)$. The low-frequency approximation of f at scale a is $Lf(a, b) = f * \check{\phi}_a(b)$ and it can be shown that:

$$f(x) = \frac{1}{C_\psi a_0} Lf(a_0, \cdot) * \phi_{a_0}(x) + \frac{1}{C_\psi} \int_0^{a_0} Wf(a, \cdot) * \psi_a(x) \frac{\mathrm{d}a}{a^2}$$

From the CWT to the DWT

- We need to discretize the CWT for numerical applications
- It requires to choose a sampling grid, that is a **discrete lattice**

$$\Gamma = \{a_j, b_{j,k}, j, k \in \mathbb{Z}\}$$

- Noting $\psi_{j,k} = \psi_{a_j, b_{j,k}}$ and $\tilde{\psi}_{j,k}$ explicitely derived from $\psi_{j,k}$ we want:

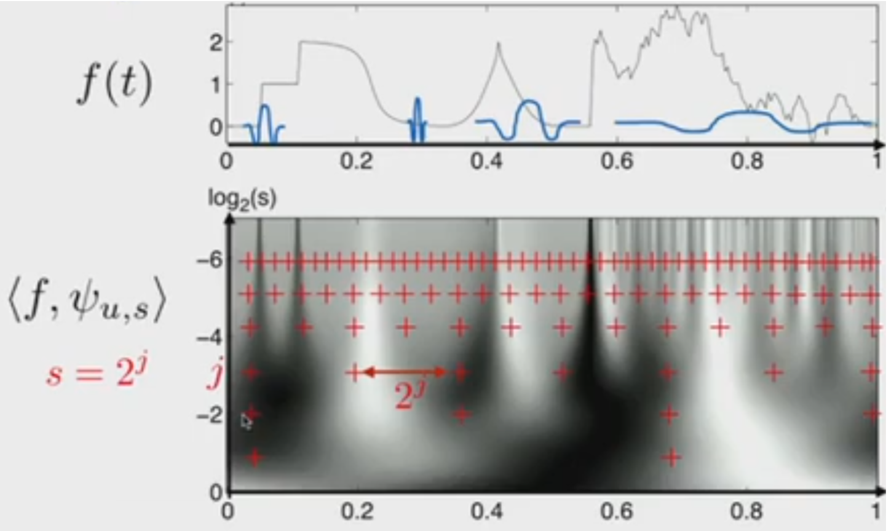
$$f = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \tilde{\psi}_{j,k}$$

- The dyadic grid corresponds to the choice $a_j = 2^{-j}$ and $b_{j,k} = k2^{-j}$

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j, k \in \mathbb{Z}$$

$\Rightarrow$ mostly leads to frames not bases.

From the CWT to the DWT



Credits: S. Mallat

From the CWT to the DWT

Definition (Frame)

$\{\psi_{j,k}\}$ is is a frame in the Hilbert space $\mathcal{H}$ if there exists $B \geqslant A > 0$ such that

$$A\|f\|^2 \leqslant \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 \leqslant B\|f\|^2$$

- A, B are the frame bounds
- $A = B \neq 1$ is a tight frame
- $A = B = 1$ and $\|\psi_{j,k}\| = 1$ is an orthonormal basis

$\Rightarrow$ Given a wavelet ψ we need to find lattice Γ such that $\{\psi_{j,k}\}$ is a "good frame" that is $\frac{A}{B} \approx 1$.

From frames to bases

Questions

- Can we reconstruct any function of Hilbert space from the discrete subset of wavelet coefficients?
- Is there a basis of orthogonal wavelets on $L^2(\mathbb{R})$?
- How can we construct such wavelets? With specific properties: regular, with compact support, ...
- Is there a fast algorithm to compute them?

The effervescence

- **Meyer** made the link with the Calderon's identity

$$f(x) = \int_0^{+\infty} \int_{\mathbb{R}} Wf(a,b)\psi_{a,b}(x)\,\mathrm{d}b\frac{\mathrm{d}a}{a^2}$$

- **Meyer**, **Grossmann**, **Daubechies** (1985): construction of $L^2(\mathbb{R})$ bases:

$$f(x) = \sum_{j,k} d_{j,k} 2^{j/2}\psi(2^j x - k)$$

- **Meyer**, **Malat** (1986): Fast Wavelet Transform (FWT)

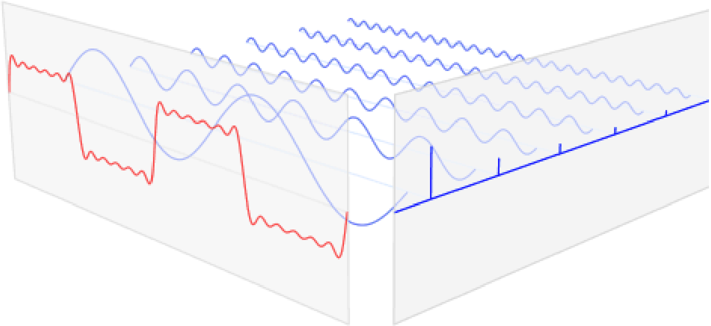
Yves Meyer

Ingrid Daubechies

Stéphane Mallat

Fourier series limitations

Discontinuities require a lot of sinusoids to be described



$$f(x) = \begin{cases} -1 \text{ if } -\pi \leqslant x < 0 \\ +1 \text{ if } 0 \leqslant x < \pi \end{cases} = \sum_{n=1}^{+\infty} \frac{4}{\pi(2n-1)} \sin((2n-1)x)$$

Credits: Wikipedia (https://en.wikipedia.org/wiki/Fourier_series)

From Fourier series to Wavelet series

$$f(x) = \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} d_{j,k} 2^{j/2} \psi(2^j x - k)$$

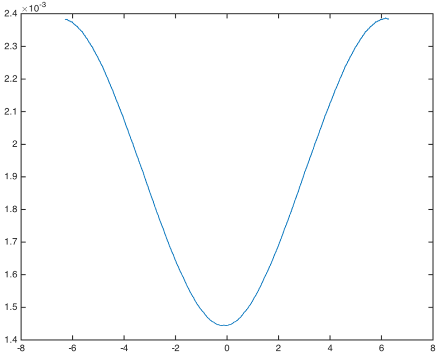


Figure: For $J = 0$ the approximation contains $N = 1$ terms

From Fourier series to Wavelet series

$$f(x) = \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} d_{j,k} 2^{j/2} \psi(2^j x - k)$$

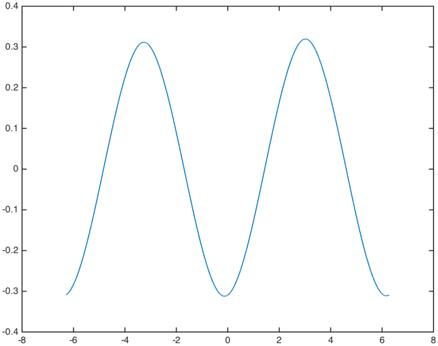


Figure: For $J = 0$ the approximation contains $N = 1 + 2$ terms

From Fourier series to Wavelet series

$$f(x) = \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} d_{j,k} 2^{j/2} \psi(2^j x - k)$$

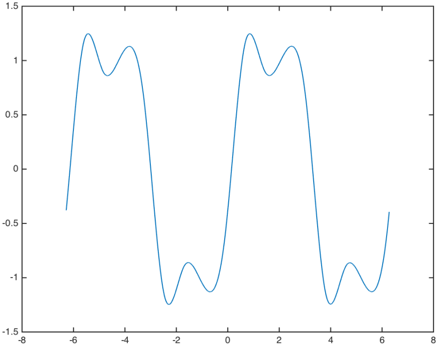


Figure: For $J = 0$ the approximation contains $N = 1 + 2 + 4$ terms

From Fourier series to Wavelet series

$$f(x) = \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} d_{j,k} 2^{j/2} \psi(2^j x - k)$$

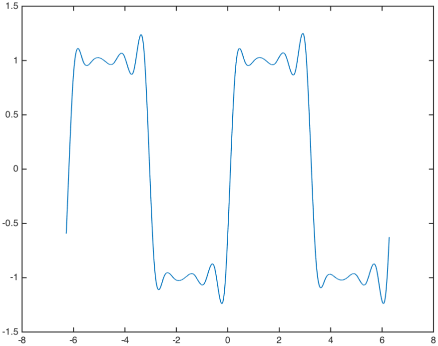


Figure: For $J = 0$ the approximation contains $N = 1 + 2 + 4 + 8$ terms

From Fourier series to Wavelet series

$$f(x) = \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} d_{j,k} 2^{j/2} \psi(2^j x - k)$$

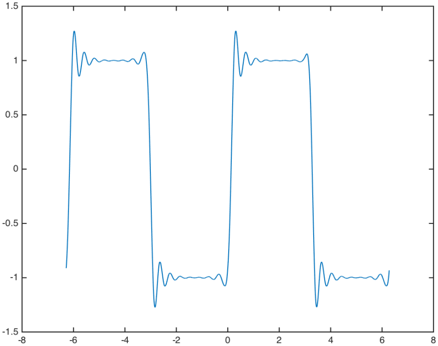


Figure: For $J = 0$ the approximation contains $N = 1 + 2 + 4 + 8 + 16$ terms

From Fourier series to Wavelet series

$$f(x) = \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} d_{j,k} 2^{j/2} \psi(2^j x - k)$$

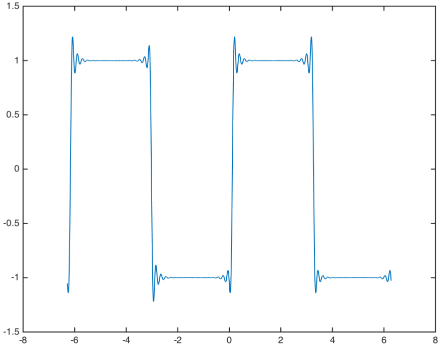


Figure: For $J = 0$ the approximation contains $N = 1 + 2 + 4 + 8 + 16 + 32$ terms

From Fourier series to Wavelet series

$$f(x) = \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} d_{j,k} 2^{j/2} \psi(2^j x - k)$$

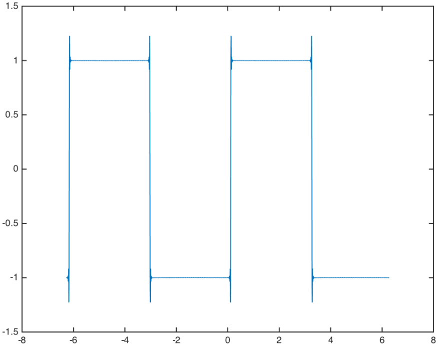


Figure: For $J = 0$ the approximation contains $N = 1 + \ldots + 512 = 1023$ terms

From Fourier series to Wavelet series

$$f(x) \approx \sum_{|d_{j,k}|>10^{-2}} d_{j,k} 2^{j/2} \psi(2^j x - k)$$

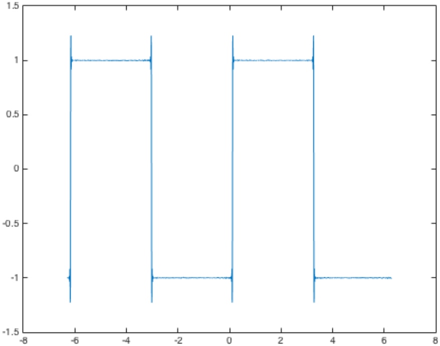


Figure: The approximation contains $N = 207$ terms

The four musketeers of wavelets

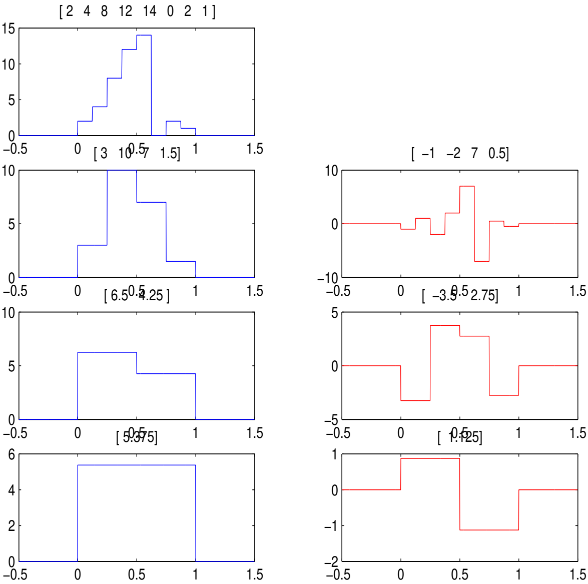
Figure: Stéphane Mallat, Yves Meyer, Ingrid Daubechies & Emmanuel Candès

1. The Haar Basis

Decomposition algorithm

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad \mathbf{[2 \quad 4 \quad 8 \quad 12 \quad 14 \quad 0 \quad 2 \quad 1]} \\
\qquad\qquad\quad \downarrow \text{(means)} \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \text{(details)} \\
\qquad\quad \mathbf{[3 \quad 10 \quad 7 \quad 1.5]} \qquad\qquad\qquad\qquad \color{red}{\mathbf{[-1 \quad -2 \quad 7 \quad 0.5]}} \\
\qquad\quad \downarrow \text{(means)} \qquad \downarrow \text{(details)} \\
\qquad \mathbf{[6.5 \quad 4.25]} \qquad\quad \color{red}{\mathbf{[-3.5 \quad 2.75]}} \\
\downarrow \text{(means)} \quad \downarrow \text{(details)} \\
\color{blue}{\mathbf{[5.375]}} \qquad \color{red}{\mathbf{[1.125]}} \\
\color{magenta}{\Longleftrightarrow} \\
\mathbf{[}\color{blue}{\mathbf{5.375}} \qquad \color{red}{\mathbf{1.125 \quad -3.5 \quad 2.75 \quad -1 \quad -2 \quad 7 \quad 0.5]}}
\end{array}
$$

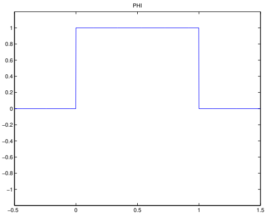
Decomposition algorithm

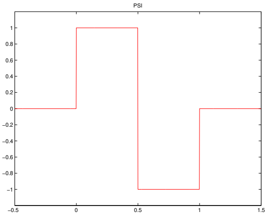


Credits: V. Perrier

The Haar basis of $L^2(0,1)$

Let $\varphi = 1$ on $[0,1]$ and $\psi(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}[\\ -1 & \text{if } x \in [\frac{1}{2}, 1[\end{cases}$





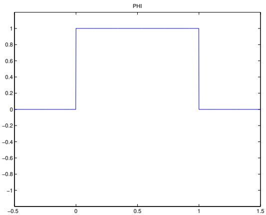
For $j \geq 0$ and $0 \leq k \leq 2^j - 1$, one set: $\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$ then

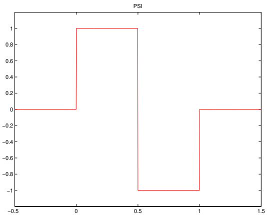
$$\psi_{j,k}(x) = \begin{cases} 2^{\frac{j}{2}} & \text{if } x \in [k2^{-j}, (k+\frac{1}{2})2^{-j}[\\ -(2^{\frac{j}{2}}) & \text{if } x \in [(k+\frac{1}{2})2^{-j}, (k+1)2^{-j}[\end{cases}$$

The family $\{\varphi, \psi_{j,k}\}$ is an **orthonormal basis** of $L^2(0,1)$, called Haar basis.

The Haar basis of $L^2(0,1)$

Let $\varphi = 1$ on $[0,1]$ and $\psi(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}[\\ -1 & \text{if } x \in [\frac{1}{2}, 1[\end{cases}$





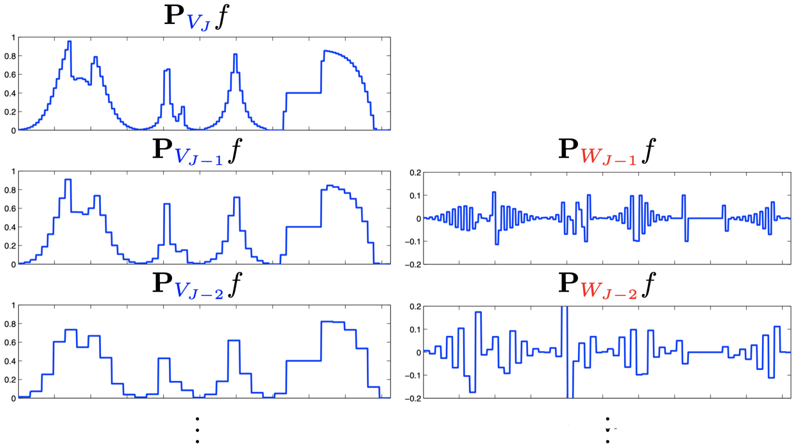
For $j \geq 0$ and $0 \leq k \leq 2^j - 1$, one set: $\varphi_{j,k}(x) = 2^{\frac{j}{2}}\varphi(2^j x - k)$ then

$$\varphi_{j,k}(x) = \begin{cases} 2^{\frac{j}{2}} & \text{if } x \in [k2^{-j}, (k+1)2^{-j}[\\ 0 & \text{otherwise} \end{cases}$$

Compression: $\varphi_{j,k} = \dfrac{\varphi_{j+1,2k} + \varphi_{j+1,2k+1}}{\sqrt{2}}$, $\psi_{j,k} = \dfrac{\varphi_{j+1,2k} - \varphi_{j+1,2k+1}}{\sqrt{2}}$

The Haar basis of $L^2(0,1)$

- Projection on approx. space: $\mathbf{P}_{V_j} f = \sum_k \langle f, \varphi_{j,k} \rangle \varphi_{j,k} = \sum_k c_{j,k} \varphi_{j,k}$
- Projection on details space: $\mathbf{P}_{W_j} f = \sum_k \langle f, \psi_{j,k} \rangle \psi_{j,k} = \sum_k d_{j,k} \psi_{j,k}$
- Projection on orthogonal spaces: $\mathbf{P}_{V_{j+1}} f = \mathbf{P}_{V_j} f + \mathbf{P}_{W_j} f$



Credits: G. Peyré

The Haar basis of $L^2(0,1)$

- V_j: vector space of constant functions on $\left\{\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right]\right\}_{k=0,..,2^j-1}$
- The family $\psi_{j,k}(t)$ defines a o.n.b of W_j (dim $2^j - 1$) such that

$$V_{j+1} = {\color{blue}V_j} \oplus {\color{red}W_j}$$

- $f^{j+1}(x) = \mathbf{P}_{V_{j+1}} f(x) = \sum_{k=0}^{2^{j+1}-1} c_{j+1,k}\varphi_{j+1,k}(x)$
- $f^{j+1}(x) = \mathbf{P}_{\color{blue}V_j} f(x) + \mathbf{P}_{\color{red}W_j} f(x) = \sum_{k=0}^{2^j-1} {\color{blue}c_{j,k}}\varphi_{j,k}(x) + \sum_{k=0}^{2^j-1} {\color{red}d_{j,k}}\psi_{j,k}(x)$

$$\begin{tikzcd}
 & \mathbf{P}_{V_j} f \arrow[dr] & \\
\mathbf{P}_{V_{j+1}} f \arrow[ur] \arrow[dr] & & + \mathbf{P}_{V_{j+1}} f \\
 & \mathbf{P}_{W_j} f \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{\langle f, \varphi_{j,k}\rangle\}_k \arrow[dr] & \\
\{\langle f, \varphi_{j+1,k}\rangle\}_k \arrow[ur] \arrow[dr] & & + \{\langle f, \varphi_{j+1,k}\rangle\}_k \\
 & \{\langle f, \psi_{j,k}\rangle\}_k \arrow[ur] &
\end{tikzcd}$$

- ${\color{blue}c_{j,k}} = \dfrac{c_{j+1,2k} + c_{j+1,2k+1}}{\sqrt{2}}$, ${\color{red}d_{j,k}} = \dfrac{c_{j+1,2k} - c_{j+1,2k+1}}{\sqrt{2}}$

The Haar basis of $L^2(0,1)$

- **Decompression**:

$$\varphi_{j+1,2k} = \frac{\varphi_{j,k} + \psi_{j,k}}{\sqrt{2}}, \quad \varphi_{j+1,2k+1} = \frac{\varphi_{j,k} - \psi_{j,k}}{\sqrt{2}}$$

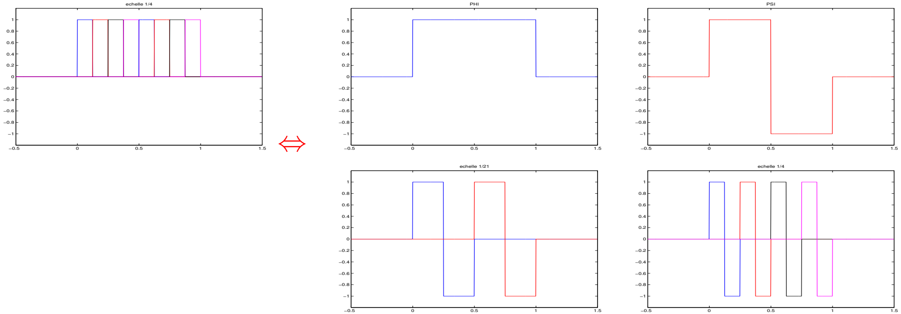
- $f^{j+1}(x) = \mathbf{P}_{V_{j+1}} f(x) = \displaystyle\sum_{k=0}^{2^{j+1}-1} c_{j+1,k}\varphi_{j+1,k}(x)$

- $f^{j+1}(x) = \mathbf{P}_{V_j} f(x) + \mathbf{P}_{W_j} f(x) = \displaystyle\sum_{k=0}^{2^j-1} c_{j,k}\varphi_{j,k}(x) + \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(x)$

$$\begin{tikzcd}
 & \mathbf{P}_{V_j} f \arrow[dr] & \\
\mathbf{P}_{V_{j+1}} f \arrow[ur] \arrow[dr] & & + \mathbf{P}_{V_{j+1}} f \\
 & \mathbf{P}_{W_j} f \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{\langle f, \varphi_{j,k}\rangle\}_k \arrow[dr] & \\
\{\langle f, \varphi_{j+1,k}\rangle\}_k \arrow[ur] \arrow[dr] & & + \{\langle f, \varphi_{j+1,k}\rangle\}_k \\
 & \{\langle f, \psi_{j,k}\rangle\}_k \arrow[ur] &
\end{tikzcd}$$

- $c_{j+1,2k} = \dfrac{c_{j,k} + d_{j,k}}{\sqrt{2}}, \; c_{j+1,2k+1} = \dfrac{c_{j,k} - d_{j,k}}{\sqrt{2}}$

Haar Basis Functions



Two equivalent bases of the piecewise constant function space on [0,1], associated to the subdivision $k/8, k = 0, \ldots, 7$

Credits: V. Perrier

Advantage of the decomposition

The Haar decomposition of a function $f \in L^2\,(0,1)$ finally writes:

$$f = c_0 + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} d_{j,k}\, \psi_{j,k}$$

with

$$c_0 = \langle f, \varphi \rangle = \int_0^1 f(x)\, \mathrm{d}x, \quad d_{j,k} = \langle f, \psi_{j,k} \rangle = \int_0^1 f(x)\, \psi_{j,k}(x)\, \mathrm{d}x$$

Local smoothness characterization

(i) if $f \in C^1(I_{j,k})$ then $|d_{j,k}| \leq C\, 2^{-3j/2}$

(ii) if $f \in C^\alpha(x_0)$ i.e. $|f(x) - f(x_0)| \leq k|x - x_0|^\alpha$ $(0 < \alpha < 1)$ then

$$|d_{j,k}| \leq C\, 2^{-j(\alpha+1/2)}$$

$\Rightarrow$ Useful property for compression!

Proof of (i)

For fixed $j \geq 0$ and $k \in \{0, \ldots 2^j - 1\}$, let $I_{j,k} :=]k2^{-j}, (k+1)2^{-j}[$.

$$\mathrm{Supp}\{\psi_{j,k}\} = [k2^{-j}, (k+1)2^{-j}] = \bar{I}_{j,k}$$

The Haar coefficient on $\psi_{j,k}$ of a function f is given by:

$$d_{j,k} = \int_{I_{j,k}} f \psi_{j,k}$$

If $f \in C^1(I_{j,k})$ then for all $x \in I_{j,k}$:

$$f(x) = f\left(x - \left(k + \frac{1}{2}\right)2^{-j}\right) + \left(x - \left(k + \frac{1}{2}\right)2^{-j}\right) f'(\theta_x), \quad \theta_x \in I_{j,k}$$

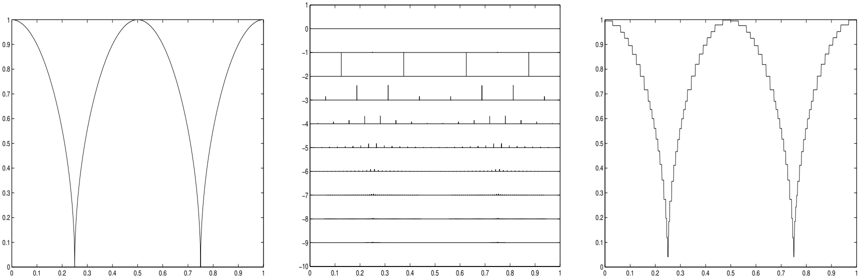
Then,

$$d_{j,k} = \int_{I_{j,k}} \left(x - \left(k + \frac{1}{2}\right)2^{-j}\right) f'(\theta_x)\psi_{j,k}(x)\,\mathrm{d}x$$

since $\int \psi_{j,k} = 0$, hence

$$|d_{j,k}| \leq \sup_{I_{j,k}} |f'| \int_{I_{j,k}} |2^{-j-1}| 2^{j/2}\,\mathrm{d}x \leq \frac{1}{2} \sup_{I_{j,k}} |f'|\ 2^{-3j/2}$$

Example: $f(x) = \sqrt{|\cos 2\pi x|}$



Left figure: *function f sampled on* $1024 = 2^{10}$ *values.*

Middle figure: *Haar coefficient map*
(abscissa : $k2^{-j} \in [0, 1]$, ordinates: $-j$, $j = 1, \dots 9$).

Right figure: *Reconstructed function from the 80 largest coefficients (> 0.06)*
(compression = 92.2 %, L^2-relative error $= 6.10^{-3}$).

Credits: V. Perrier

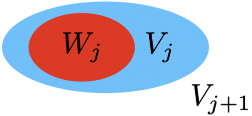
2. Regular wavelet bases

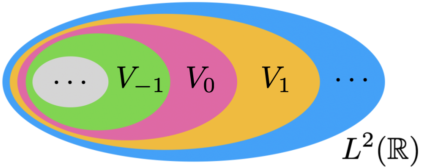
Multiresolution Analysis (MRA)

A multiresolution analysis of $L^2(\mathbb{R})$ is a sequence of closed subspaces $(V_j)_{j\in\mathbb{Z}}$ s.t.:

1. $\forall j \in \mathbb{Z},\ V_j \subset V_{j+1} \subset \cdots \to L^2(\mathbb{R})$,
2. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,
3. $f(x) \in V_j \iff f(2x) \in V_{j+1}$,
4. $f(x) \in V_0 \iff \forall n \in \mathbb{Z},\ f(x-n) \in V_0$,
5. $\exists \varphi \in V_0$ s.t $\{\varphi(x-n) : n \in \mathbb{Z}\}$ is an **orthonormal basis** of V_0.

φ is called the scaling function of the multiresolution analysis.





Multiresolution Analyses – Examples

The spaces V_j are dilation invariant, then:

$$V_j = \mathrm{Vec}\left\{\varphi_{j,k} := 2^{\frac{j}{2}}\varphi(2^j x - k)\ ;\ k \in \mathbb{Z}\right\}$$

Haar:

$$V_0 = \{\text{Piecewiese constant functions on } [k, k+1[,\ \forall k \in \mathbb{Z}\}$$

Splines of degree 1:

$$V_0 = \{\text{Continuous functions on } \mathbb{R},\ \text{affines on } [k, k+1[,\ \forall k \in \mathbb{Z}\}$$

Splines of degree n:

$$V_0 = \{\mathcal{C}^{n-1} \text{ functions on } \mathbb{R},\ \text{piecewise polynomial of deg } n \text{ on } [k, k+1[\}$$

Shannon:

$$V_0 = \{f \in L^2(\mathbb{R})\ ;\ \text{supp } \widehat{f} \subset [1, 2]\}$$

MRA – Two-scale equation for the scaling function

$V_0 \subset V_1 = \mathrm{span}\{\varphi_{1,k} := \sqrt{2}\varphi(2x-n)\ ;\ n \in \mathbb{Z}\}$, then $\varphi \in V_0$ writes:

$$\varphi(x) = \sqrt{2} \sum_{n\in\mathbb{Z}} h_n \,\varphi(2x-n) \quad \text{with} \quad h_n = \sqrt{2}\int_{\mathbb{R}} \varphi(x)\varphi(2x-n)\,\mathrm{d}x$$

Applying the Fourier Transform:

$$\widehat{\varphi}(\xi) = m_0\left(\frac{\xi}{2}\right)\widehat{\varphi}\left(\frac{\xi}{2}\right) \quad \text{with} \quad m_0(\xi) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} h_n \mathrm{e}^{-2i\pi n\xi}$$

Assume that $\varphi \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} \varphi = 1$, then:

$$\hat{\varphi}(\xi) = \prod_{j=0}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$$

(h_n) is a low pass filter and $\hat{h} = \sqrt{2}m_0$ is its **transfer function**.

MRA – Construction of the wavelets

$V_j \subset V_{j+1}$, let W_j be the orthogonal complement space of V_j in V_{j+1}:

$$V_{j+1} = V_j \oplus W_j$$

One searches for a function ψ s.t. $\{\psi(x-n) : n \in \mathbb{Z}\}$ is an **orthonormal basis** of W_0. Since $\psi \in W_0 \subset V_1$, one searches for g_n such that

$$\psi(x) = \sqrt{2} \sum_{n \in \mathbb{Z}} g_n \, \varphi(2x - n)$$

This is equivalent in Fourier domain to:

$$\widehat{\psi}(\xi) = m_1\left(\frac{\xi}{2}\right) \widehat{\varphi}\left(\frac{\xi}{2}\right) \quad \text{with} \quad m_1(\xi) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} g_n \mathrm{e}^{-2i\pi n\xi}$$

$\Rightarrow$ What are the assumptions on filters (h_n) and (g_n) in order to construct a scaling function φ and a wavelet ψ generating a MRA?

Detail filter (necessary) constraints for h

$\boxed{\Rightarrow}$ If $\{\varphi_{j,n}\}$ is an orthonormal basis of V_j then:

1. From the two-scale equation it comes

$$\widehat{h}(0) = \sqrt{2} \quad (C_1)$$

2. $\{\varphi(\cdot - n)\}_n$ orthogonal is equivalent to:

$$\forall n \in \mathbb{N}, \quad \varphi \star \check{\varphi}(n) = \delta[n] \Longleftrightarrow \sum_k |\widehat{\varphi}(\xi + 2k\pi)|^2 = 1$$

since sampling a function periodizes its Fourier transform. Inserting $\widehat{\varphi}(\xi) = 2^{-1/2}\widehat{h}(\xi/2)\widehat{\varphi}(\xi/2)$ and separating even and odd integers terms (with $\widehat{h}$ is 2π-periodic) yields:

$$\left|\widehat{h}\left(\frac{\xi}{2}\right)\right|^2 \sum_{p=-\infty}^{+\infty} \left|\widehat{\varphi}\left(\frac{\xi}{2} + 2p\pi\right)\right|^2 + \left|\widehat{h}\left(\frac{\xi}{2} + \pi\right)\right|^2 \sum_{p=-\infty}^{+\infty} \left|\widehat{\varphi}\left(\frac{\xi}{2} + \pi + 2p\pi\right)\right|^2 = 2$$

Putting $\xi' = \xi/2$ and $\xi' = \xi/2 + \pi$ in the two sums yields:

$$|\widehat{h}(\xi')|^2 + |\widehat{h}(\xi' + \pi)|^2 = 2 \quad (C_2)$$

Detail filter (sufficient) constraints for h

$\boxed{\Leftarrow}$ Conversely, the following theorem gives sufficient conditions on $\widehat{h}$ to guarantee that this infinite product is the Fourier transform of a scaling function:

Theorem (Mallat, Meyer)

If $\widehat{h}(\xi)$ is 2π-periodic and continuously differentiable in a neighborhood of $\xi = 0$, if it satisfies $(C_1), (C_2)$ and

$$\inf_{\xi \in [-\pi/2, \pi/2]} |\widehat{h}(\xi)| > 0 \quad (C_3)$$

then

$$\boxed{\widehat{\varphi}(\xi) = \prod_{p=1}^{+\infty} \frac{\widehat{h}(2^{-p}\xi)}{\sqrt{2}}}$$

Detail filter (necessary) constraints for g

$\boxed{\Rightarrow}$ If $\{\psi_{j,n}\}$ is an orthonormal basis of W_j then:

1. $\{\psi(\cdot - n)\}_n$ orthogonal is equivalent to:

$$\forall n \in \mathbb{N}, \quad \psi \star \check{\psi}(n) = \delta[n] \Longleftrightarrow \sum_k |\widehat{\psi}(\xi + 2k\pi)|^2 = 1$$

Inserting $\widehat{\psi}(\xi) = 2^{-1/2}\widehat{g}(\xi/2)\widehat{\varphi}(\xi/2)$ and separating even and odd integers terms (with $\widehat{g}$ 2π-periodic) also yields:

$$|\widehat{g}(\xi)|^2 + |\widehat{g}(\xi + \pi)|^2 = 2 \quad (C_4)$$

2. $\{\psi(\cdot - n)\}_n$ orthogonal to $\{\varphi(\cdot - n)\}_n$ is equivalent to:

$$\forall n \in \mathbb{N}, \quad \psi \star \check{\varphi}(n) = 0 \Longleftrightarrow \sum_k \widehat{\psi}(\xi + 2k\pi)\widehat{\varphi}^*(\xi + 2k\pi) = 0$$

which leads to:

$$\widehat{g}(\xi)\widehat{h}(\xi)^* + \widehat{g}(\xi + \pi)\widehat{h}(\xi + \pi)^* = 0 \quad (C_5)$$

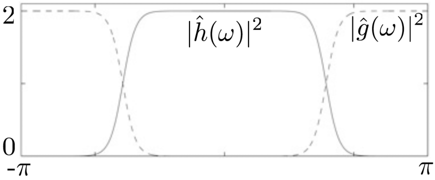
Detail filter (sufficient) constraints for g

$\boxed{\Leftarrow}$ Conversely, the following theorem gives sufficient conditions on $\widehat{h}$ and $\widehat{g}$ to guarantee that the constructed wavelets $\{\psi(\cdot - n)\}_n$ give an orthonormal basis of W_j:

Theorem (Mallat, Meyer)

Under conditions $(C_1) - (C_2) - (C_3)$

$$\{\psi(\cdot - n)\}_n \text{ orthonormal basis of } W_j \iff (C_4) + (C_5)$$



Quadrature mirror filters:

$$\widehat{g}(\xi) = \mathrm{e}^{-i2\pi\xi}\widehat{h}(\xi + \pi) \iff g[n] = (-1)^{1-n}h[1-n]$$

MRA – Wavelet decomposition

$$L^2(\mathbb{R}) = V_0 \bigoplus_{j=0}^{+\infty} W_j = \bigoplus_{j=-\infty}^{+\infty} W_j$$

$$W_j = \mathrm{Vec}\left\{\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)\ ;\ k \in \mathbb{Z}\right\}$$

Let $f \in L^2(\mathbb{R})$. Its wavelet decomposition writes:

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \varphi(x-k) + \sum_{j=0}^{+\infty} \sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(x) = \sum_{j=-\infty}^{+\infty} \sum_{k\in\mathbb{Z}} d_{j,k}\ \psi_{j,k}(x)$$

with $c_k = \langle f, \varphi(\cdot - k)\rangle$ and $d_{j,k} = \langle f, \psi_{j,k}\rangle$.

Property of wavelet bases

- Vanishing Moments. A wavelet ψ satisfies:

$$\int_{\mathbb{R}} \psi = 0$$

One usually impose N Vanishing Moments:

$$\int_{\mathbb{R}} x^n \psi = 0, \quad \forall n = 0, \dots N-1$$

- Characterization of the local smoothness of f. For $n \leq N$, $\alpha < N$,
  (i) if $f \in C^n(V_{x_0})$ then $|d_{j,k}| \leq C\ 2^{-j(n+1/2)}$ (for $k2^{-j}$ "neighbor" of x_0)

  (ii) if $f \in C^\alpha(x_0)$ i.e. $|f^{[\alpha]}(x) - f^{[\alpha]}(x_0)| \leq k|x - x_0|^{\alpha - [\alpha]}$ then

$$|d_{j,k}| \leq C\ 2^{-j(\alpha+1/2)} \quad \text{(for } k2^{-j} \text{ "neighbor" of } x_0\text{)}$$

$\Rightarrow$ Important property in view of compression

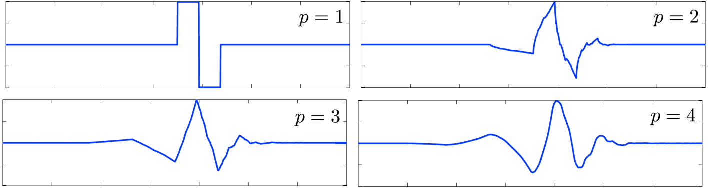
Examples of wavelets constructed from filters

Debauchies family

1. $\widehat{h}(0) = \sqrt{2}$
2. $|\widehat{h}(\xi)|^2 + |\widehat{h}(\xi + \pi)|^2 = 2$
3. p vanishing moments $\Leftrightarrow \forall k < p, \ \dfrac{\mathrm{d}^k \widehat{h}}{\mathrm{d}\xi^k}(\pi) = 0$

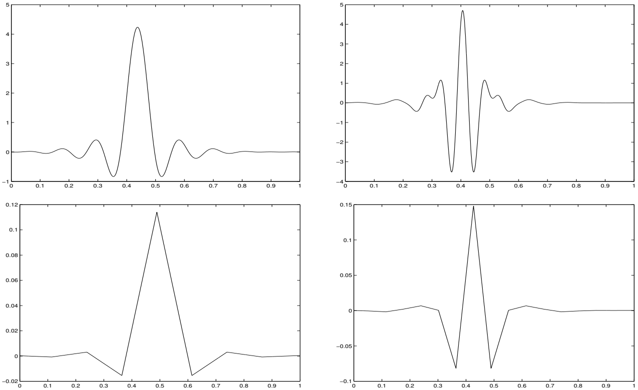
$\Rightarrow$ orthogonal wavelets with minimal support $2p - 1$

- $p = 1$ (Haar): $h = [0.7071, 0.7071]$
- $p = 2$: $h = [0.4830, 0.8365, 0.2241, -0.1294]$
- $p = 3$: $h = [0, 0.3327, 0.8069, 0.4599, -0.1350, -0.0854, 0.0352]$



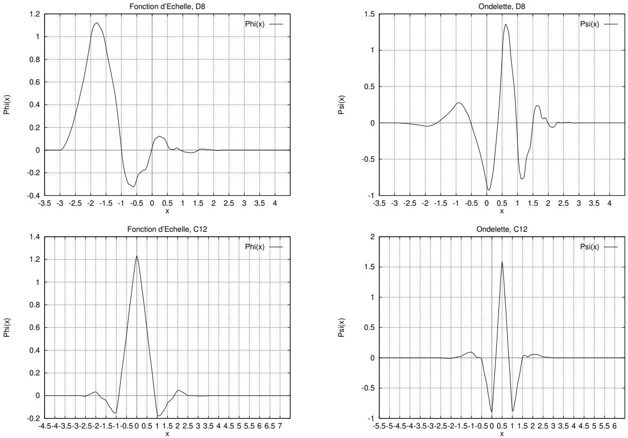
Credits: G. Peyré

Examples of scaling functions and wavelets



Scaling function (left) and wavelet (right):
1st line: Meyer functions (C^∞ and infinite number of vanishing moments).
2d line: Splines of degree 1 (2 vanishing moments).

Examples of scaling functions and wavelets



Scaling function (left) and wavelet (right) compactly supported:
1st line: D8 (4 vanishing moments).
2d line: Coifman C12 (4 vanishing moments).

Fast Wavelet transform (FWT)

Let f be a discrete 1D signal discret 1D of length $N = 2^J$.
Step 0 of the algorithm: computing the coefficients $c_J = (c_{J,k})$

$$c_{J,k} \approx 2^{-\frac{J}{2}} f(k2^{-J}), \quad k \in \mathbb{Z},$$

(using $\int \varphi = 1$). One consider the function f_J of V_J:

$$f_J = \sum_{k \in \mathbb{Z}} c_{Jk} \varphi_{Jk}$$

Decomposition: $V_J = V_0 \oplus W_0 \oplus \cdots \oplus W_{J-1}$
For $j = J, \ldots, 1$ one uses $V_j = V_{j-1} \oplus W_{j-1}$, and then $\forall k \in \mathbb{Z}$:

$$c_{j-1,k} = \sum_{n \in \mathbb{Z}} c_{j,n} h_{n-2k}$$

$$d_{j-1,k} = \sum_{n \in \mathbb{Z}} c_{j,n} g_{n-2k}$$

Fast Wavelet transform (FWT)

Proof: From the two-scale equation

$$\varphi(x) = \sqrt{2} \sum_{n\in\mathbb{Z}} h_n \, \varphi(2x - n)$$

by replacing $x \leftarrow 2^j x - k$ and multiplying by $2^{j/2}$ we get:

$$\begin{aligned}
2^{j/2}\varphi(2^j x - k) \quad &= \quad 2^{j/2}\sqrt{2}\sum_{n\in\mathbb{Z}} h_n \varphi(2^{j+1}x - (n+2k)) \\
\varphi_{j,k}(x) \quad &\overset{n\leftarrow n-2k}{=} \quad \sum_{n\in\mathbb{Z}} h_{n-2k}\varphi_{j+1,n} \\
c_{j,k} \quad &\overset{\langle f,\cdot\rangle}{=} \quad \sum_{n\in\mathbb{Z}} h_{n-2k} c_{j+1,n}
\end{aligned}$$

Example (Haar wavelets)

- Low-pass filter: $h = [\cdots, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \cdots]$
- High-pass filter: $g = [\cdots, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \cdots]$

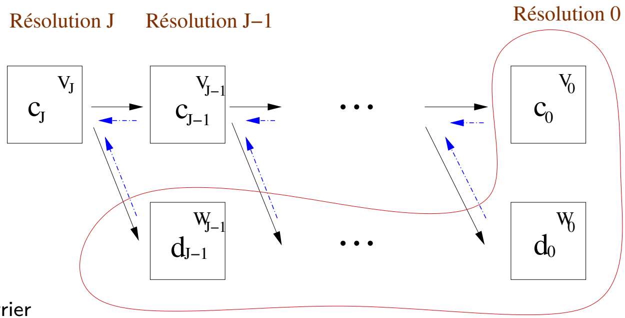
Fast Wavelet transform (FWT)

Noting $c_j = (c_{j,k})_{k \in \mathbb{Z}}$:

<u>convolution - decimation:</u>

$$c_{j-1}[k] = (c_j \star \check{h})[2k], \quad \forall k \in \mathbb{Z}$$
$$d_{j-1}[k] = (c_j \star \check{g})[2k], \quad \forall k \in \mathbb{Z}$$

with $\check{h}[n] = h[-n]$ and $\check{g}[n] = g[-n]$.



Credits: V. Perrier

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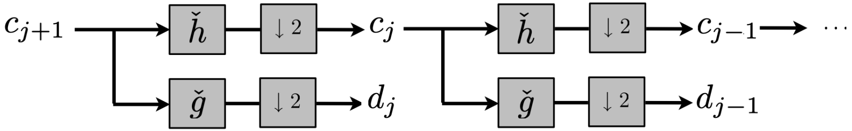
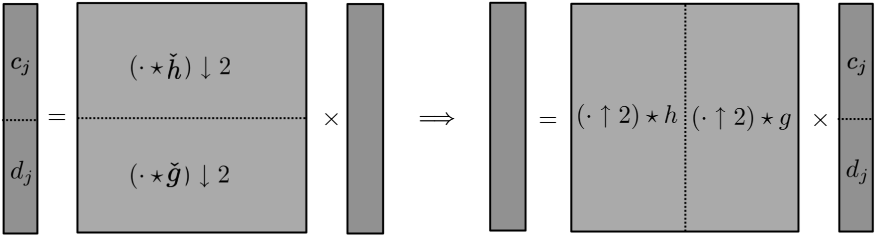


Credits: V. Perrier

Fast Wavelet transform (FWT)

Recomposition: From the wavelet coefficients and the scaling coefficients at scale 0: $[c_{0k}, \{d_{jk}\}_{j=0\cdots J-1, k\in\mathbb{Z}}]$, one wants to retrieve the scaling coefficients at finest scale J: $c_J = [(c_{Jk})_{k\in\mathbb{Z}}]$.



One uses $V_{j-1} \oplus W_{j-1} = V_j$, for $j = 0, \ldots, J-1$:

$$c_{j,k} = \sum_{n\in\mathbb{Z}} c_{j-1,n} h_{k-2n} + \sum_{n\in\mathbb{Z}} d_{j-1,n} g_{k-2n}, \quad \forall k \in \mathbb{Z}$$

Fast Wavelet transform (FWT)

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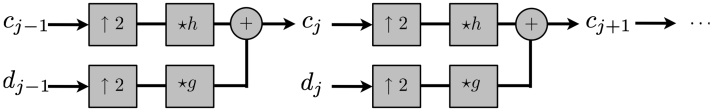
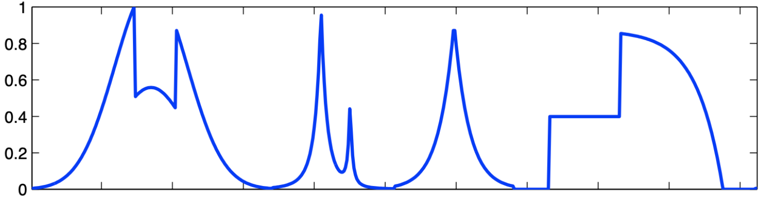
It writes, in vector form, noting:

$$\widetilde{x_n} = (x \uparrow 2)[n] = \begin{cases} x_p & \text{if } n = 2p \\ 0 & \text{if } n = 2p+1 \end{cases}$$

$$\boxed{c_j[k] = (\widetilde{c_{j-1}} \star h)[k] + (\widetilde{d_{j-1}} \star g)[k]}$$



Fast Wavelet Transform algorithm

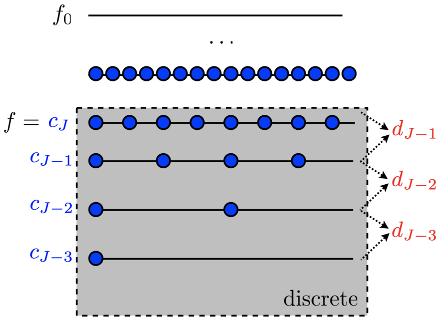


Algorithm (FWT)

Initialization: $c_J = f$, $N = 2^J$

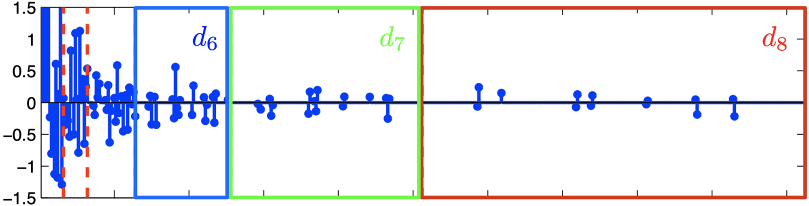
For $j = J, \ldots, 0$

$$c_{j-1} = (c_j \star \check{h}) \downarrow 2$$
$$d_{j-1} = (c_j \star \check{g}) \downarrow 2$$



Credits: G. Peyré

Fast Wavelet Transform algorithm



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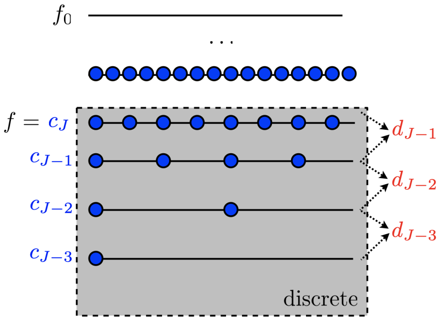
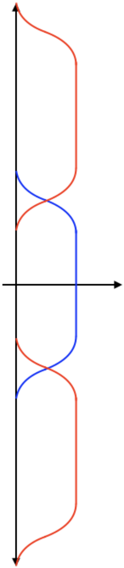
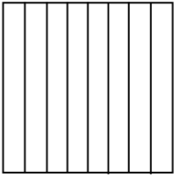


Credits: G. Peyré

Fast Wavelet Transform algorithm

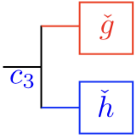
$\xrightarrow{c_3}$

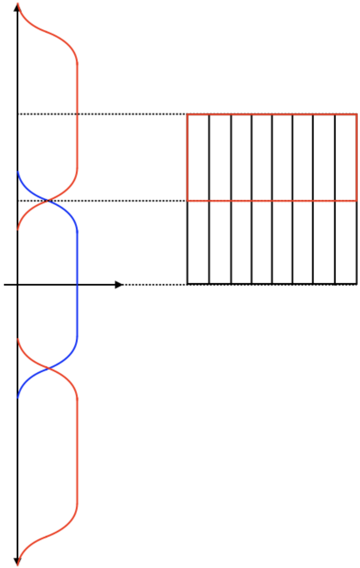




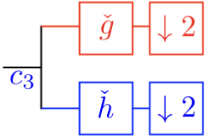
c_3

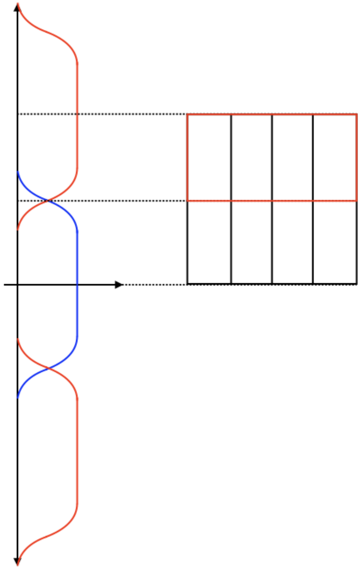
Fast Wavelet Transform algorithm



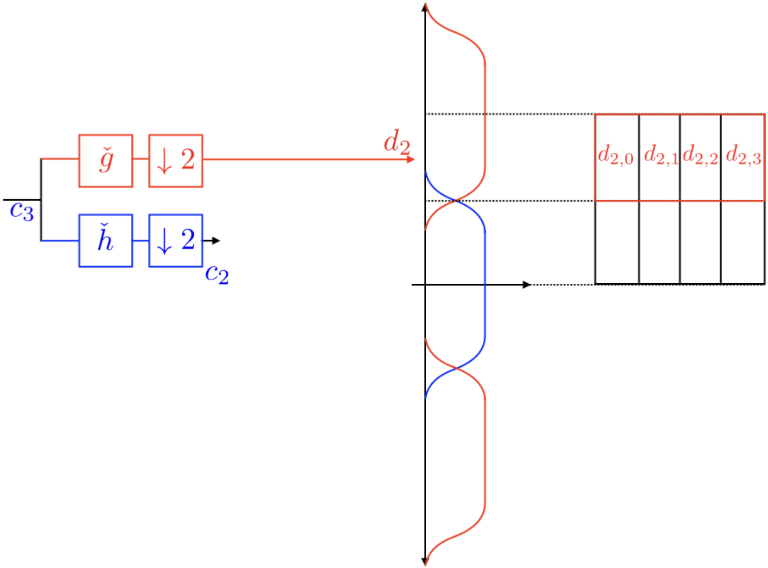


Fast Wavelet Transform algorithm

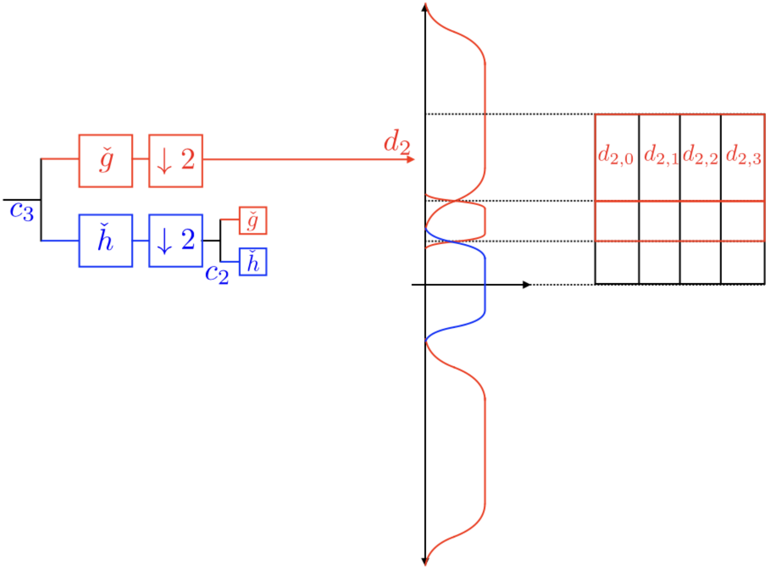




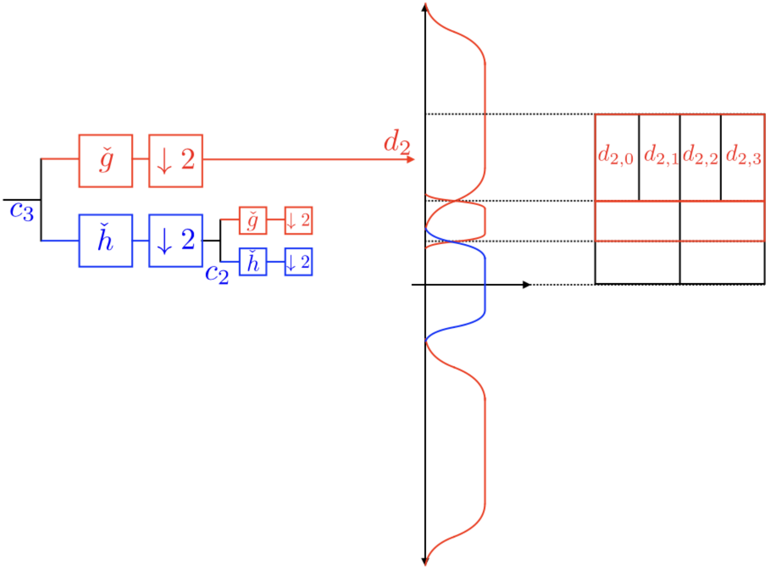
Fast Wavelet Transform algorithm



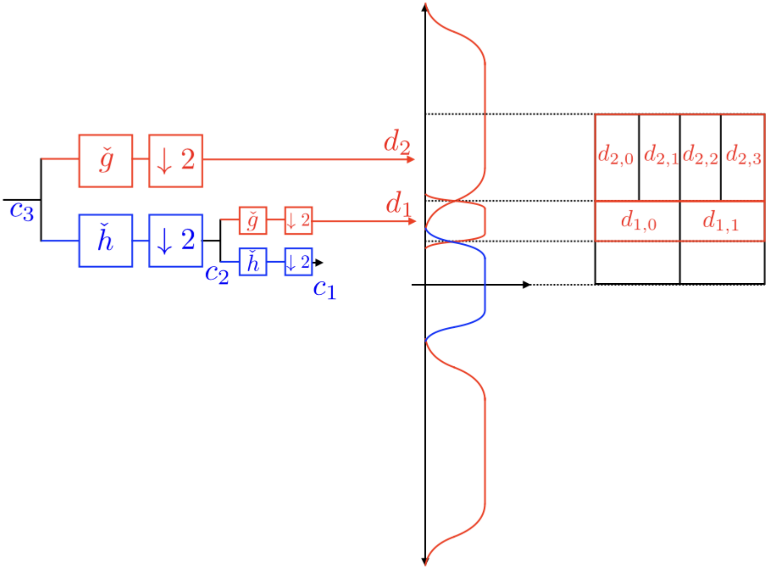
Fast Wavelet Transform algorithm



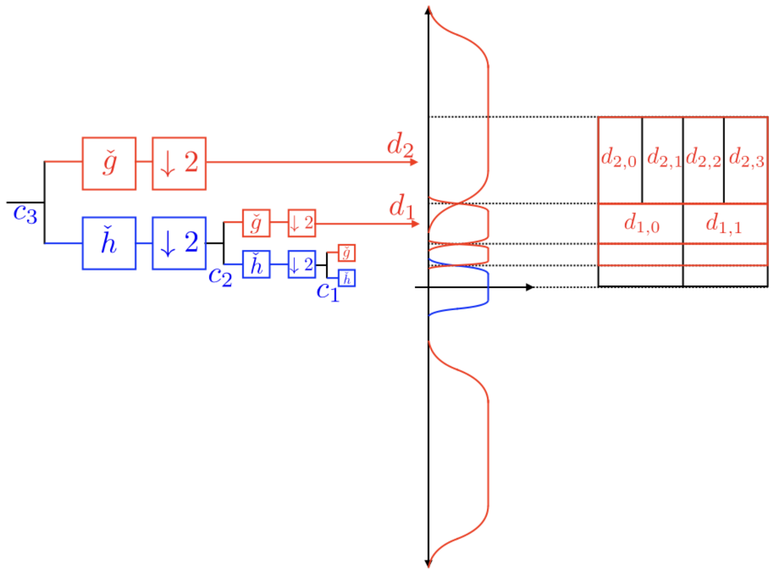
Fast Wavelet Transform algorithm



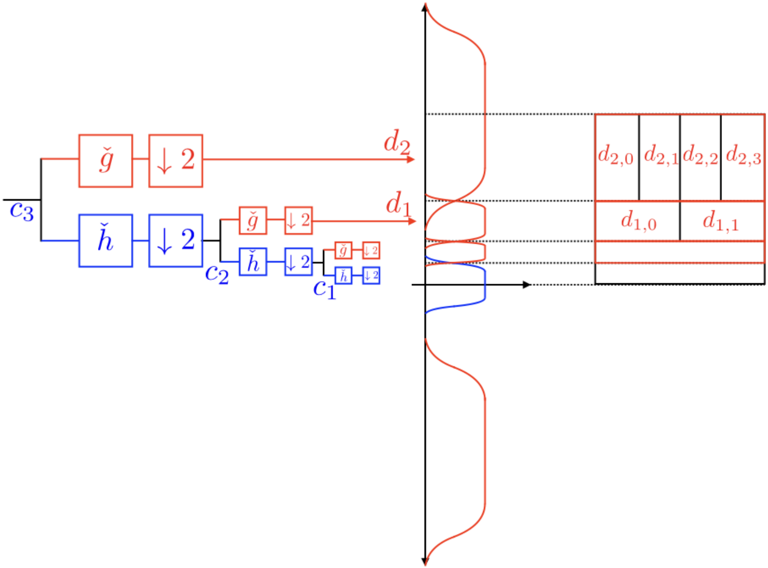
Fast Wavelet Transform algorithm



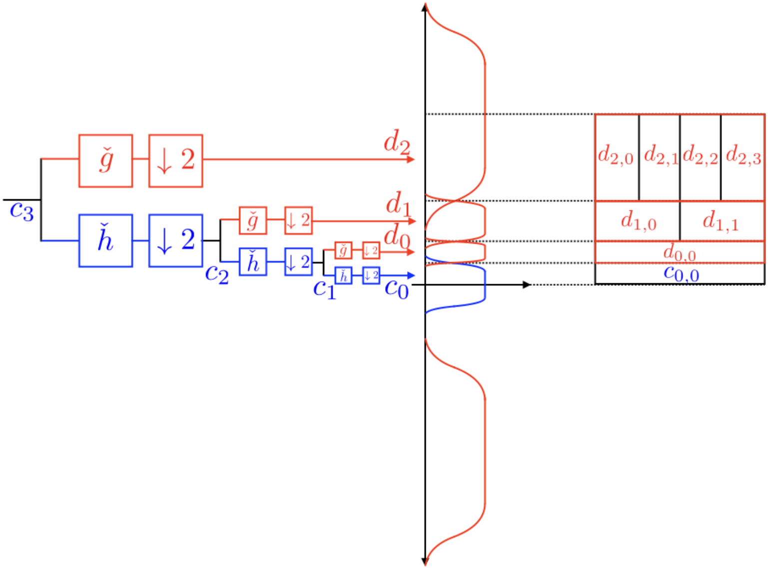
Fast Wavelet Transform algorithm



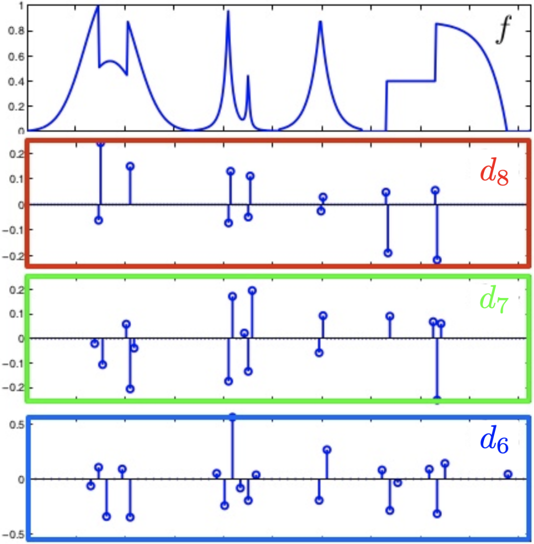
Fast Wavelet Transform algorithm



Fast Wavelet Transform algorithm

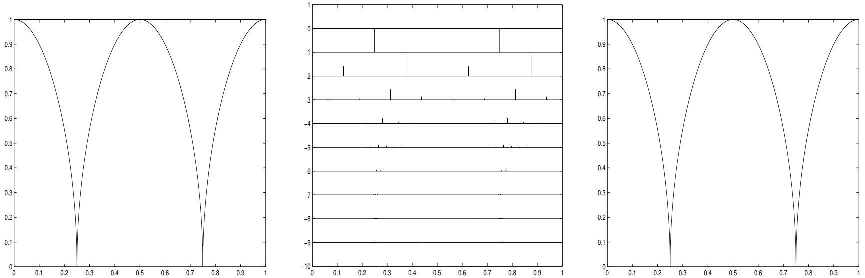


Example from Mallat



Credits: G. Peyré

Example: $f(x) = \sqrt{|\cos 2\pi x|}$



Left figure : *function f discretized on* $1024 = 2^{10}$ *values.*

Middle figure: *wavelet coefficient map D8*
(abscissa: $k2^{-j} \in [0, 1]$*, ordinate:* $-j$*,* $j = 1, \ldots 9$*).*

Right figure: *reconstructed function with the 80 highest coefficients* ($> 10^{-3}$)
(compression = 92.2 %, relative error $L^2 = 3.10^{-7}$*).*

Credits: V. Perrier