Wavelets and Applications

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Linear time-invariant filtering in classical signal processing

- **Time-invariant operator** L. If the input $f(t)$ is *delayed/shifted* by τ, $f_\tau(t) = f(t-\tau)$, then the output is also *delayed/shifted* by τ:

$$g(t) = Lf(t) \Rightarrow g(t-\tau) = Lf_\tau(t)$$

- **Impulse response** h of L:

$$h(t) = L\delta(t) \Rightarrow h(t-\tau) = L\delta_\tau$$

Proposition

A time-invariant linear filtering L is equivalent to a convolution with the impulse response h.

Proof. Assume that f is continuous so that $f(t) = \int_{-\infty}^{\infty} f(\tau)\delta_\tau(t)\,\mathrm{d}\tau$ and L is linear and *(weak)* continuous hence

$$\begin{aligned} Lf(t) &= \int_{-\infty}^{\infty} f(\tau) L\delta_\tau(t)\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,\mathrm{d}\tau = (f*h)(t) \qquad \square \end{aligned}$$

Linear time-invariant filtering in classical signal processing

- With $f[k] = f(kT)$, the sampled signal is

$$f_s(t) = \sum_{k=-\infty}^{\infty} f[k]\delta(t - kT)$$

- Let $g(t) = f(t - kT)$ then

$$g[n] = g(nT) = f(nT - kT) = f((n-k)T) = f[n-k] = (\mathcal{T}_k f)[n]$$

Proposition

A time-invariant linear filtering L is equivalent to a convolution with the impulse response h.

Proof.

$$(Lf_s)(nT) = \sum_{k=-\infty}^{\infty} f[k]L(\delta[n-k]) = \sum_{k=-\infty}^{\infty} f[k]h[n-k] = (f \star h)[n] \qquad \square$$

Diagonalization of time-invariant operators

Proposition

Complex exponentials are eigenvectors of convolution operators

Proof. Consider $f(t) = \mathrm{e}^{i2\pi\xi t}$ and the sampled signal $f_s(t)$

- *Continuous case:*

$$L\,\mathrm{e}^{i2\pi\xi t} = \int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{i2\pi\xi(t-\tau)}\,\mathrm{d}\tau = \mathrm{e}^{i2\pi\xi t}\int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{-i2\pi\xi\tau}\,\mathrm{d}\tau = \hat{h}(\xi)\,\mathrm{e}^{i2\pi\xi t}$$

- *Discrete case:*

$$L\,\mathrm{e}^{i2\pi\xi nT} = \sum_{k=-\infty}^{\infty} h[k]\mathrm{e}^{i2\pi\xi(n-k)T} = \mathrm{e}^{i2\pi\xi nT}\sum_{k=-\infty}^{\infty} h[k]\mathrm{e}^{-i2\pi\xi kT} = H(\xi)\,\mathrm{e}^{i2\pi\xi nT}$$

NB. The Fourier transform of $\delta(t-kT)$ is $\mathrm{e}^{-i2\pi\xi kT}$, which lead to the **Discrete Time Fourier Transform** (DTFT):

$$H(\xi) = \sum_{k=-\infty}^{\infty} h[k]\mathrm{e}^{-i2\pi\xi kT} = \mathcal{F}\left(\sum_{k=-\infty}^{\infty} h[k]\delta(t-kT)\right) = \hat{f}_s(\xi)$$

The $\mathcal{Z}$-transform

More generally notice that for a complex z:

$$L\, z^n = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} = H(z)\, z^n$$

which involves the linear $\mathcal{Z}$-transform of h defined as:

$$\mathcal{Z}(\{h_k\}_k) : z \in \mathbb{C} \mapsto H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

- $H(z)$ is the transfer function of L
- DTFT corresponds to the z-transform evaluated in $z = \mathrm{e}^{i2\pi\xi T}$

Properties

- *Translation:* $\mathcal{Z}(\mathcal{T}_l h)(z) = \mathcal{Z}(\{h_{k-l}\}_k)(z) = z^{-l} H(z)$
- *Scaling:* $\mathcal{Z}(\mathcal{D}_a h)(z) = \mathcal{Z}(a^k \{h_k\}_k)(z) = H\left(\frac{z}{a}\right)$
- *Convolution:* $\mathcal{Z}(h_1 * h_2)(z) = \mathcal{Z}(\{\sum_l h_{1,l} h_{2,k-l}\}_k)(z) = H_1(z) H_2(z)$

FIR filters

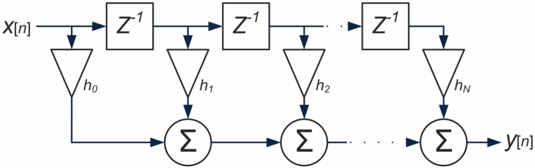
Definition

Let a filter with an impulse response h. The filter is said to be with a *Finite Impulse Response* (FIR) if h is **finite** that is $h = \{h_n\}_{n=0}^{N}$ and

$$H(z) = \sum_{n=0}^{N} h_n z^{-n} \quad (\text{polynomial in } z^{-1})$$

▶ The FIR filter **difference equation** for a discrete time signal f, output g and filter coefficients h at sample k is:

$$g[k] = (f \star h)[k] = h[0]f[k] + h[1]f[k-1] + \cdots + h[N]f[k-N]$$



The role of shift operator in classical signal processing

Let consider the (formal) polynomial representation of the signal f *via* its $\mathcal{Z}$-transform:

$$F(z) = \sum_{n=0}^{N-1} f[n] z^{-n}$$

as well for the output $g = Lf$ represented by $G(z)$. Then we have:

$$G(z) = F(z)H(z) \Longleftrightarrow H(z) = \frac{G(z)}{F(z)} \quad \text{(transfert function)}$$

Shift operator

Let consider a periodic extension of f where the real line is folded around the circle $f_n = f_{n \bmod N}$ and let define the *shift operator* which perform a simple delay

$$f = [f_0, f_1, \dots, f_{N-1}] \mapsto g = \mathrm{shift}\, f = [f_{N-1}, f_0, \dots, f_{N-1}]$$

It is clear that $H_{\mathrm{shift}}(z) = z^{-1}$. Observe the **shift invariance** with any other operator L due to the commutativity $z^{-1} \cdot H_L(z) = H_L(z) \cdot z^{-1}$.

Shift operator on graphs

Analogy between 1-D periodic signal and the ring graph

The (directed) ring graph, associated to a periodic time-serie $\mathbf{f} = [f_0, f_2, \ldots, f_{n-1}]\top \in \mathbb{R}^n$ with $f[k+n] = f[k]$, has the following adjacency matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

Notice that $\mathbf{g} = \mathbf{Af} = [f_{n-1}, f_0, \ldots, f_{n-2}]\top$ is the signal $\mathbf{f}$ shifted by one.

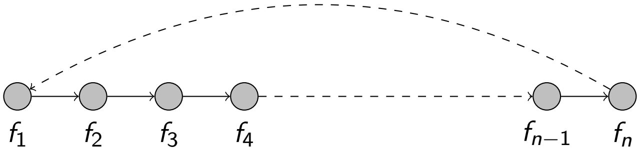


Figure: A (directed) ring graph

Filtering operator on graphs

- Using the matricial notation, a filter h on a graph can be in general represented by the matrix $\mathbf{H}$:

$$\mathbf{g} = \mathbf{H}\mathbf{f}$$

- Let $\mathbf{A}$ be an arbitrary adjacency matrix, which play the role of the *shift operator* on neighbors. Following the analogy with classical signal processing, a filter represented by $\mathbf{H}$ is said to be shift-invariant if it commutes with the shift, that is:

$$\mathbf{A}\mathbf{H} = \mathbf{H}\mathbf{A}$$

- If the characteristic and minimal polynomial of $\mathbf{A}$ are equals then every filter commuting with $\mathbf{A}$ is a polynomial in $\mathbf{A}$ i.e

$$\mathbf{H} = H(\mathbf{A}) = \sum_{k=0}^{K} h_k \mathbf{A}^k$$

Diagonalization of shift-invariant operators on graph

Proposition

The eigenvectors of the shift operator $\mathbf{A}$ are the eigenfunctions of the polynomial filter $\mathbf{H}$.

Proof. Let consider $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}$ where $\mathbf{U} = (\mathbf{u}_1|\cdots|\mathbf{u}_n)$ are the eigenvectors and $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the matrix of eigenvalues of $\mathbf{A}$. Then, it is straightforward to verify that:

$$\mathbf{H} = H(\mathbf{A}) = \sum_{m=0}^{M} h_m(\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1})^m = \mathbf{U}H(\boldsymbol{\Lambda})\mathbf{U}^{-1}$$

where $H(\boldsymbol{\Lambda}) = \mathrm{diag}(H(\lambda_1), \ldots, H(\lambda_n))$. Finally one has

$$\mathbf{H}\mathbf{u}_k = \mathbf{U}H(\boldsymbol{\Lambda})\mathbf{U}^{-1}\mathbf{u}_k = \mathbf{U}H(\boldsymbol{\Lambda})\mathbf{e}_k = H(\lambda_k)\mathbf{u}_k$$

$\Rightarrow$ invariance of the eigenvectors of the shift operator $\mathbf{A}$ with respect to graph filters.

Frequency analysis of graph signals

Analogy with the 1D Fourier transform

- In the 1-D continuous time setting one can decompose a signal on the Fourier basis of complex exponentials:

$$f(t) = \int_{-\infty}^{+\infty} \hat{f}(\xi) \mathrm{e}^{2\pi i t \xi} \mathrm{d}\xi$$

  where the argument $2\pi\xi$ determines the frequency of oscillation of such functions.

- Observe that this basis also corresponds to the eigenfunctions of the 1-D **Laplace operator** $\Delta = -\frac{\partial^2}{\partial t^2}$:

$$-\frac{\partial^2}{\partial t^2} \mathrm{e}^{2\pi i t \xi} = (2\pi\xi)^2 \, \mathrm{e}^{2\pi i t \xi}$$

$\Rightarrow$ One can consider the Laplacian matrix $\mathbf{L}$ as the *shift operator* instead of the adjacency matrix $\mathbf{A}$.

Frequency analysis of graph signals

Analogy between 1-D periodic signal and the ring graph

The **undirected** ring graph has the following adjacency matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

$$\mathbf{L} = \mathbf{D} - \mathbf{A} = 2\mathbf{I} - \mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

Circulant matrices and Fourier basis

- Any circulant matrix $\mathbf{C}$ is **diagonalizable** in the Fourier basis $\mathbf{C} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top$ where

$$\mathbf{U} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega & \dots & \omega^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \dots & \omega^{(n-1)^2} \end{pmatrix}$$

  and $\omega = \exp\left(\frac{2\pi i}{n}\right)$ is a primitive n-th root of unity.
- The columns $\mathbf{u}_k$ (*Fourier modes*) of the matrix $\mathbf{U}$ are the eigenvectors of any circulant matrix.
- Multiplying a vector $\mathbf{f} \in \mathbb{R}^n$ by $\mathbf{U}$ performs a discrete Fourier transform (DFT)

$$\widehat{\mathbf{f}} = \mathbf{U}^\top \mathbf{f} = \begin{pmatrix} \langle \mathbf{u}_1, \mathbf{f} \rangle \\ \vdots \\ \langle \mathbf{u}_n, \mathbf{f} \rangle \end{pmatrix}$$

Graph Fourier basis

Equivalence with classical Fourier basis for the ring graph

- The **undirected** ring graph has this *circulant* Laplacian matrix:

$$\mathbf{L} = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

- The eigendecomposition of the graph Laplacian is

$$\mathbf{L} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top$$

  where the eigenvectors $\mathbf{u}_k$ are the Fourier modes.

- The eigenvalues are given by

$$\lambda_k = 2 - 2\cos\frac{\pi k}{n}$$

Graph reference matrix

Definition

Suppose we have a graph reference matrix $\mathbf{R}$ associated to the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$ as:

$$\forall i \neq j, \quad \mathbf{R}_{ij} \neq 0 \Leftrightarrow (i \to j) \in \mathcal{E}$$

and suppose it is diagonalizable in $\mathbb{C}$ as:

$$\mathbf{R} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}$$

with $\mathbf{U} = (\mathbf{u}_1 | \cdots | \mathbf{u}_n)$ and $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$

Graph Fourier modes

Definition

The eigenvectors $\{\mathbf{u}_k\}$ of $\mathbf{R}$ are considered to be graph Fourier modes and $\{\lambda_k\}$ their associated graph frequency if:

1. (consistency) $\mathbf{R}$ is circulant for $\mathcal{G}$ reduced to the ring graph
2. (variational interpretation) $\mathrm{Re}(\lambda_k)$ or $|\lambda_k|$ is a measure of variation of $\mathbf{u}_k$

Laplacian matrix

For undirected graphs $\mathbf{R} \leftarrow \mathbf{L} = \mathbf{D} - \mathbf{A}$ satisfies the two properties.

Many others possible definitions!

- $\mathbf{R} \leftarrow \mathbf{L}_n = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ (*normalized Laplacian*)
- $\mathbf{R} \leftarrow \mathbf{L}_d = \mathbf{I} - \frac{\mathbf{A}}{\lambda_{\max(\mathbf{A})}}$ (*deformed Laplacian*)
- $\mathbf{R} \leftarrow \mathbf{L}_{rw} = \mathbf{I} - \mathbf{D}^{-1}\mathbf{A}$ (*random walk Laplacian*)

Graph Fourier transform

The case of the normalized Laplacian of undirected graphs

Let consider the **graph reference matrix**:

$$\mathbf{R} \leftarrow \mathbf{L}_n = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top$$

- $\mathbf{U} = (\mathbf{u}_1|\cdots|\mathbf{u}_n)$ is a graph Fourier basis, each $\mathbf{u}_k$ is a generalized (co)sine
- $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ the spectrum of $\mathbf{L}_n$, each λ_k is a generalized (squared) frequency and

$$0 = \lambda_1 \leqslant \lambda_2 \leqslant \lambda_n \leqslant 2$$

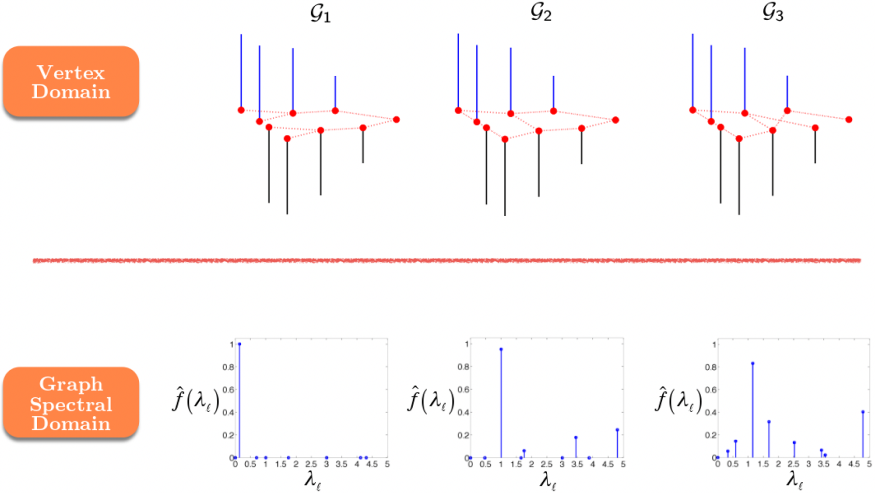
- Variational interpretation:

$$\lambda_k = \mathbf{u}_k^\top \mathbf{L}_n \mathbf{u}_k = \frac{1}{2} \sum_{i \sim j} \mathbf{A}_{ij} \left[\frac{\mathbf{u}_k(i)}{\sqrt{d_i}} - \frac{\mathbf{u}_k(j)}{\sqrt{d_j}} \right]^2$$

- The Graph Fourier transform of a graph signal $\mathbf{f} \in \mathbb{R}^n$ reads

$$\widehat{\mathbf{f}} = \mathbf{U}^\top \mathbf{f}$$

The graph Fourier transform encodes the graph structure

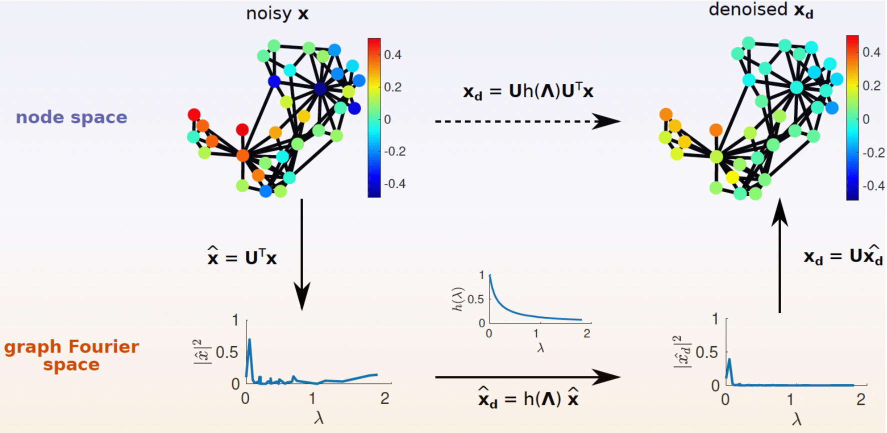


Credits: D. Shuman

Graph filtering

Given a filter H defined in the Fourier space, the signal $\mathbf{f}$ filtered by h is

$$\mathbf{g} = \mathbf{U} H(\mathbf{\Lambda}) \mathbf{U}^\top \mathbf{f}$$



Credits: N. Tremblay

Fast Graph filtering

- **Problem**: Computing $\mathbf{g} = \mathbf{U} H(\mathbf{\Lambda}) \mathbf{U}^\top \mathbf{f}$ costs $\mathcal{O}(n^3)$
- **Solution**: To use a polynomial approximation of order p of h:

$$\tilde{H}(\lambda) = \sum_{l=1}^{p} \alpha_l \lambda^l \approx H(\lambda)$$

Indeed, in this case one has:

$$\mathbf{g} = \mathbf{U} H(\mathbf{\Lambda}) \mathbf{U}^\top \mathbf{f} \approx \mathbf{U} \tilde{H}(\mathbf{\Lambda}) \mathbf{U}^\top \mathbf{f} = \mathbf{U} \sum_{l=1}^{p} \alpha_l \Lambda^l \mathbf{U}^\top \mathbf{f} = \sum_{l=1}^{p} \alpha_l \mathbf{L}^l \mathbf{f}$$

$\rightsquigarrow$ only involves matrix-vector multiplication of cost $\mathcal{O}(p|\mathcal{E}|)$

Credits: N. Tremblay

Generalized convolution on graphs

In the vertex domain, the n-th element of the output signal

$$\mathbf{g} = \mathbf{U} H(\mathbf{\Lambda}) \mathbf{U}^\top \mathbf{f}$$

is given by this kind of **generalized convolution**[1] on the graph:

$$g(i) = (f * h)(i) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f(j) \mathbf{u}_k(j) H(\lambda_k) \mathbf{u}_k(i) = \sum_{j=0}^{n-1} f(j) h_i(j)$$

where the transfert function is defined by:

$$H(\lambda_k) = h_0 + h_1 \lambda_k + \cdots + h_M \lambda_k^M$$

and the graph impulse response is:

$$h_i(j) = \sum_{k=0}^{n-1} H(\lambda_k) \mathbf{u}_k(i) \mathbf{u}_k(j) = (h * \delta_i)(j) = \frac{1}{\sqrt{n}} (\mathcal{T}_i h)(j)$$

[1] By replacing the complex exponentials $\psi_k(t) = \mathrm{e}^{2\pi i k t}$ by Laplacian eigenvectors $\mathbf{u}_k(n)$ in the classical relationship $g(t) = (f * h)(t) = \int_{\mathbb{R}} \hat{g}(k) \psi_k(t) \, \mathrm{d}k = \int_{\mathbb{R}} \hat{f}(k) \hat{g}(k) \psi_k(t) \, \mathrm{d}k$.

Tikhonov regularization

- We observe a **noisy** graph signal $\mathbf{y} = \mathbf{f}_0 + \boldsymbol{\epsilon}$ where $\boldsymbol{\epsilon}$ is uncorrelated additive Gaussian noise, and we want to recover $\mathbf{f}_0$ which is a smooth with respect to the underlying graph.

- To enforce this *a priori* information we penalize the optimization problem with a **regularization term** of the form $\mathbf{f}^\top \mathbf{L} \mathbf{f}$ measuring the smoothness and a fixed γ:

$$J(\mathbf{f}) = \frac{1}{2}\|\mathbf{f} - \mathbf{y}\|_2^2 + \gamma\, \mathbf{f}^\top \mathbf{L} \mathbf{f} \mathbf{f}^\star = \underset{\mathbf{f} \in \mathbb{R}^n}{\arg\min} \|\mathbf{f} - \mathbf{y}\|_2^2 + \gamma\, \mathbf{f}^\top \mathbf{L} \mathbf{f}$$

- The **optimal reconstruction** is given by

$$\mathbf{f}^\star = H(\mathbf{L})\mathbf{y}, \quad H(\lambda) = \frac{1}{1 + 2\gamma\lambda}$$

Proof. We want to minimize the objective function

$$J(\mathbf{f}) = \frac{1}{2}\|\mathbf{f} - \mathbf{y}\|_2^2 + \gamma\, \mathbf{f}^\top \mathbf{L}\mathbf{f}$$

By differentiating

$$\frac{\partial J}{\partial \mathbf{f}} = \mathbf{f} - \mathbf{y} + 2\gamma \mathbf{L}\mathbf{f} = 0$$

which results in

$$\mathbf{f} = (\mathbf{I} + 2\gamma \mathbf{L})^{-1}\mathbf{y}$$

In the spectral domain, from $\mathbf{L} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top$ and by noting $\mathbf{Y} = \mathbf{U}^\top \mathbf{y}$ and $\mathbf{F} = \mathbf{U}^\top \mathbf{f}$ one finally has

$$\mathbf{F} = (\mathbf{I} + 2\gamma \boldsymbol{\Lambda})^{-1}\mathbf{Y}$$

hence the expression

$$H(\lambda) = \frac{1}{1 + 2\gamma\lambda}$$

$\square$

Windowed Graph Fourier Transform

- **Modulation operator** for a function $f \in L^2(\mathbb{R})$ is defined by

$$(\mathcal{M}_\xi f)(t) = \mathrm{e}^{2\pi i \xi t} f(t)$$

- Let $g \in L^2(\mathbb{R})$ a window, the **windowed Fourier atom** is given by

$$g_{u,\xi}(t) = (\mathcal{M}_\xi \mathcal{T}_u g)(t) = g(t-u)\mathrm{e}^{2\pi i \xi t}$$

- By analogy one can define the generalized modulation operator by:

$$(\mathcal{M}_k f)(i) = \sqrt{n} f(i) \mathbf{u}_k(i)$$

- Then a **windowed graph Fourier atom** by:

$$g_{i,k}(j) = (\mathcal{M}_k \mathcal{T}_i g)(j) = N \mathbf{u}_k(j) \sum_{\ell=0}^{n-1} G(\lambda_\ell) \mathbf{u}_\ell(i) \mathbf{u}_\ell(j)$$

- The **windowed graph Fourier transform** by:

$$Sf(i,k) = \langle f, g_{i,k} \rangle$$

Wavelets on graph

- The 1D continuous wavelet transform of $f \in L^2(\mathbb{R})$ at scale a and position b is given by:

$$Wf(a,b) = \frac{1}{a}\int_{\mathbb{R}} \psi^*\left(\frac{x-b}{a}\right) f(x)\,\mathrm{d}x = \int_{\mathbb{R}} \hat{\psi}^*(a\xi)\hat{f}(\xi)\mathrm{e}^{i2\pi\xi b}\,\mathrm{d}\xi$$

- The wavelet at scale a centered around b reads:

$$\psi_{a,b}(x) = \int_{\mathbb{R}} \hat{\psi}^*(a\xi)\hat{\delta}_b(\xi)\mathrm{e}^{i2\pi\xi x}\,\mathrm{d}\xi$$

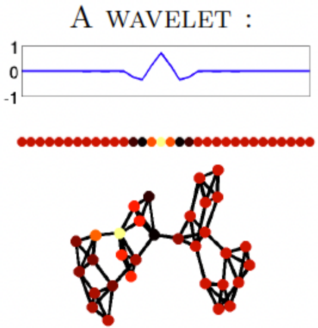
▶ By analogy one can define the wavelet transform of f at node i of the graph and scale $a > 0$ by:

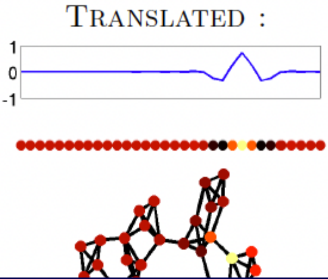
$$Wf(i,a) = \sum_{k=0}^{n-1} H(a\lambda_k)\hat{f}(\lambda_k)\mathbf{u}_k(i)$$

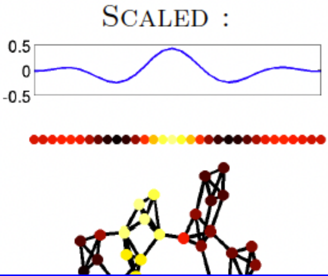
▶ The wavelet on graph is defined as:

$$\psi_{a,b} = \mathbf{U}H(a\mathbf{\Lambda})\mathbf{U}^\top \boldsymbol{\delta}_a$$

Wavelets on graph

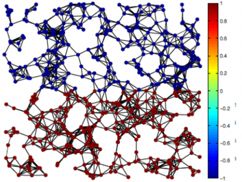




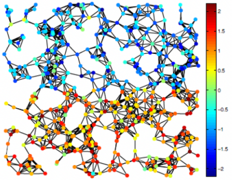


Some applications of wavelets on graph

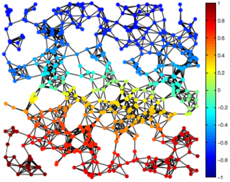
- Tikhonov regularization for denoising : $\mathrm{argmin}_f\{\|f - y\|_2^2 + \gamma f^\top \mathrm{L} f\}$



Original

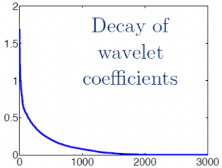


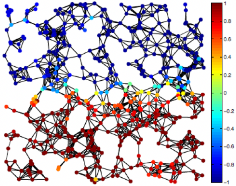
Noisy



Denoised

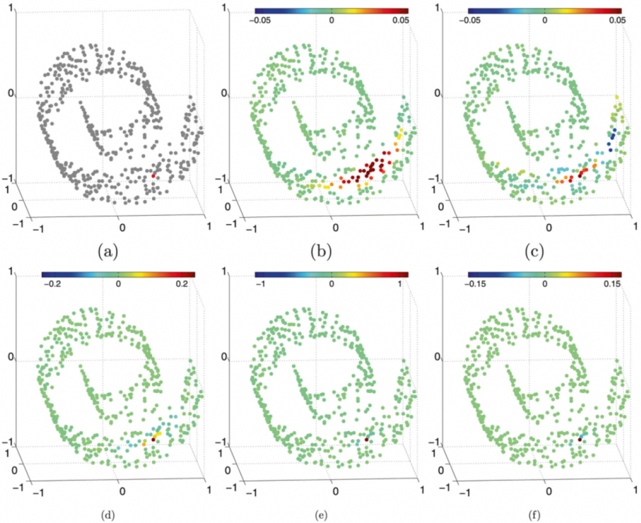
- Wavelet denoising : $\mathrm{argmin}_a\{\|y - \mathrm{W}^* a\|_2^2 + \gamma \|a\|_1\}$





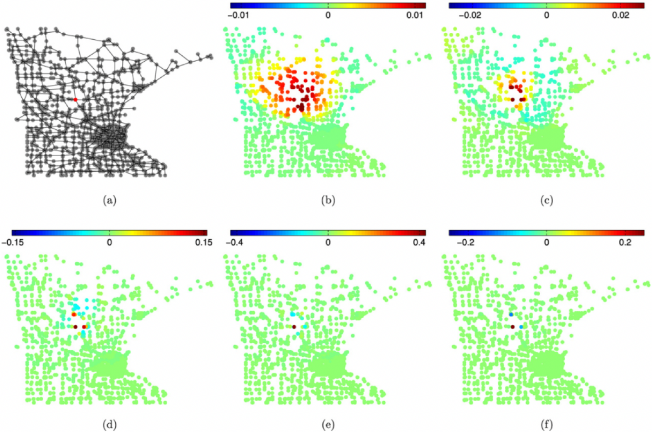
Figures courtesy of D. Shuman

Some applications of wavelets on graph



Figures from Hammond et al., *Wavelets on graphs via spectral graph theory*, 2011

Some applications of wavelets on graph



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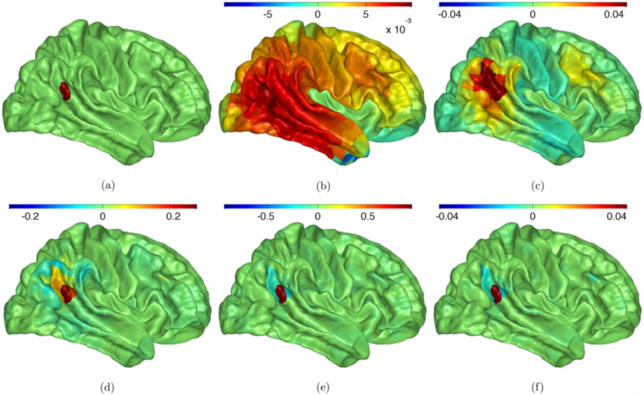

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