Convex Super-Resolution Detection of Lines in Images

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joint work with

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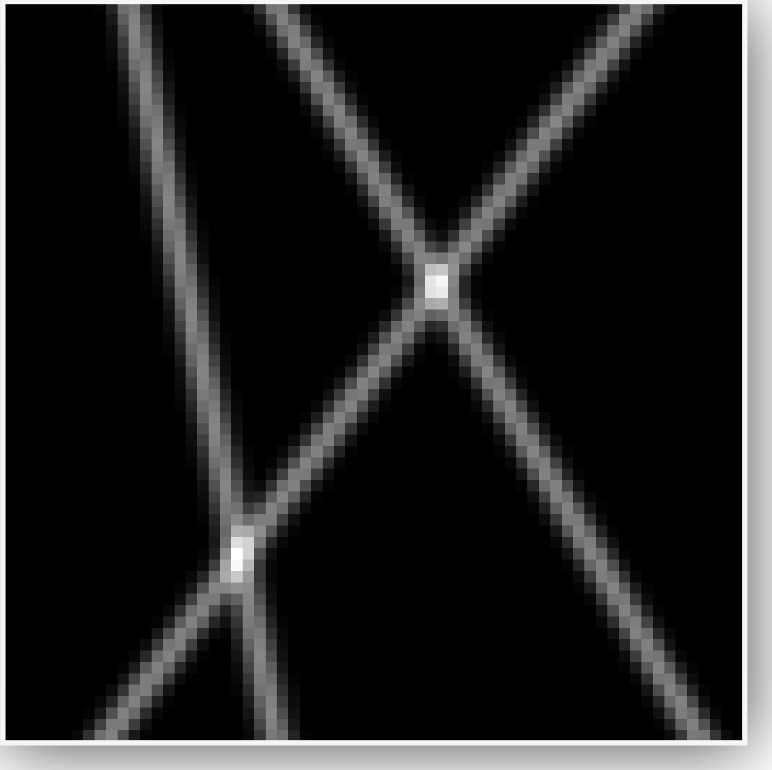

Workshop ATLAS : Apprentissage, optimisation à large échelle et calcul distribué.

Outline

- Introduction
  - Motivation
  - Super-Resolution principle
- Problem formulation
  - Ideal and blurred model of lines
  - Framework of atomic norm minimization
- Reconstruction methods
  - Solving the optimization problem by primal-dual algorithm
  - Recovering the line parameters by Prony method
- Numerical experiments
- Conclusion and future work

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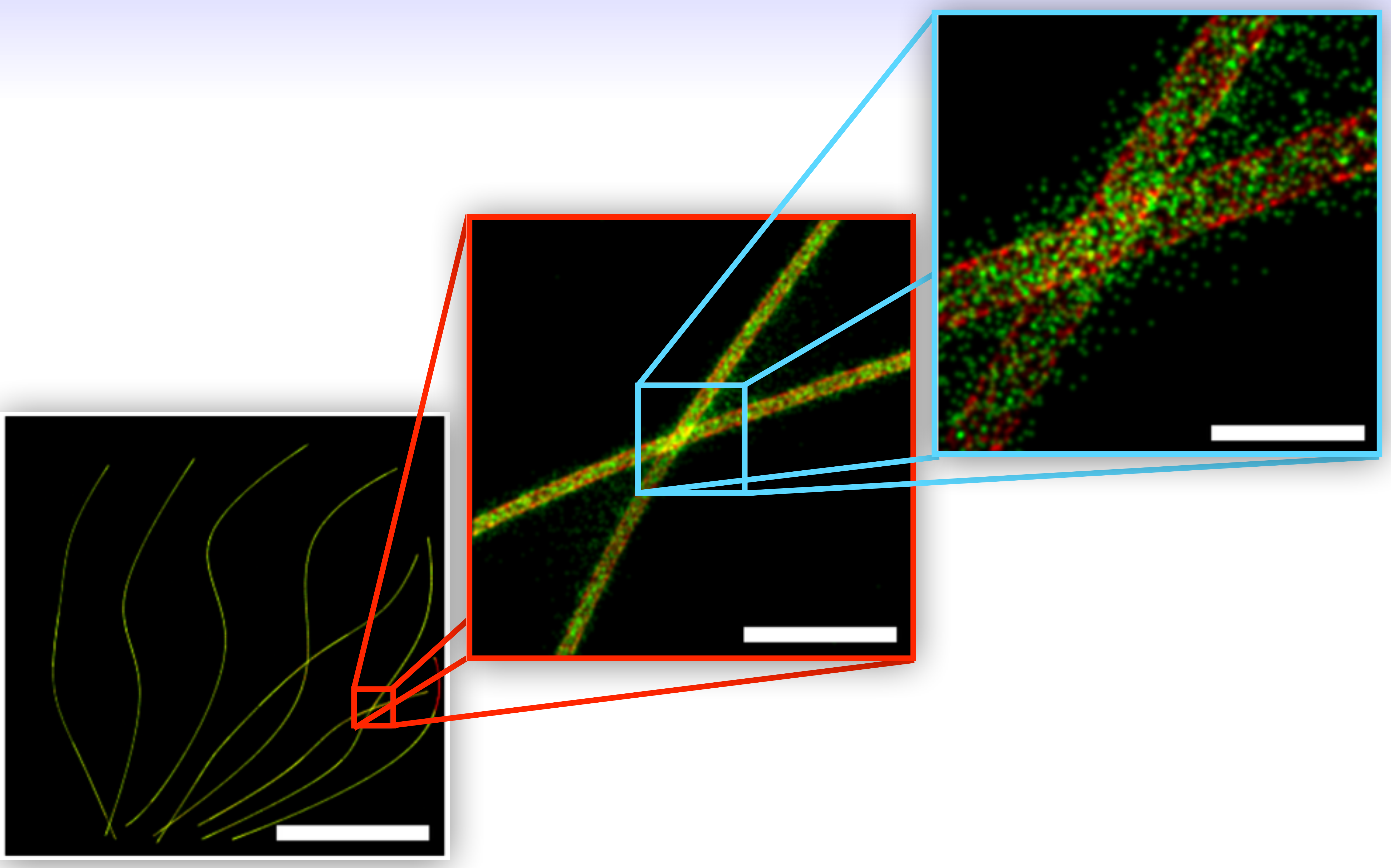
Motivation



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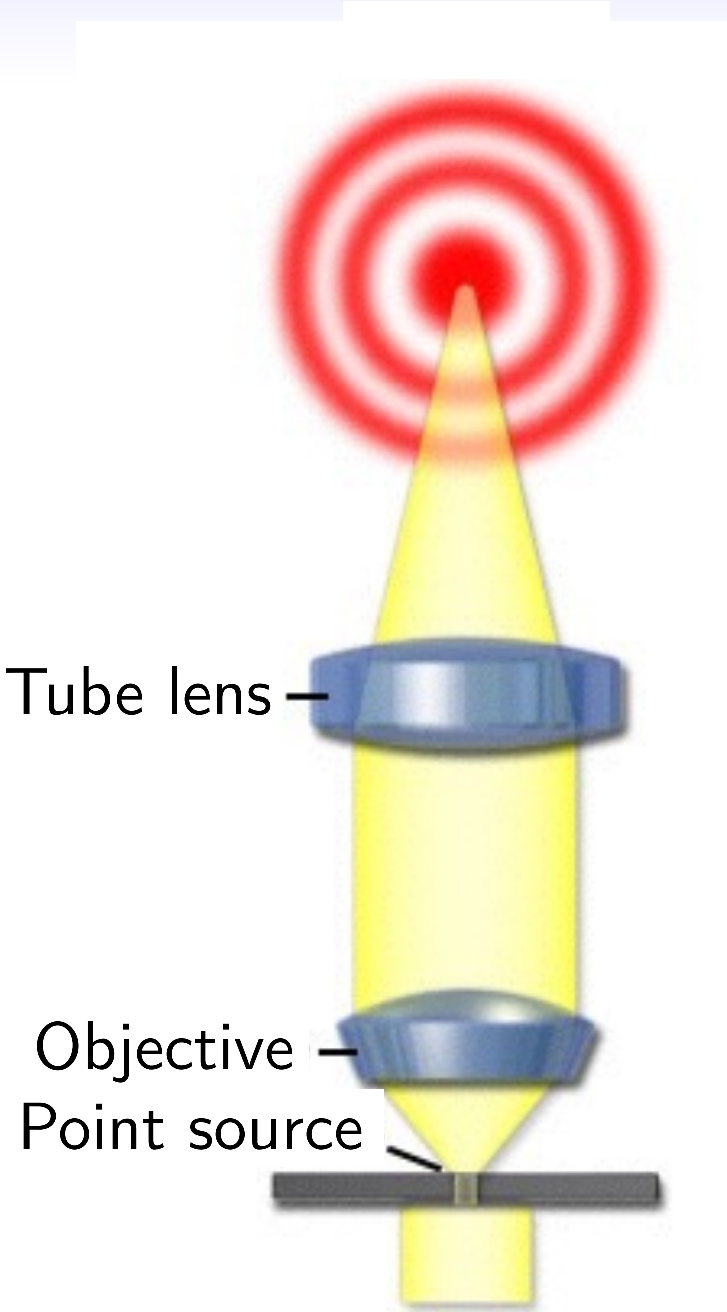
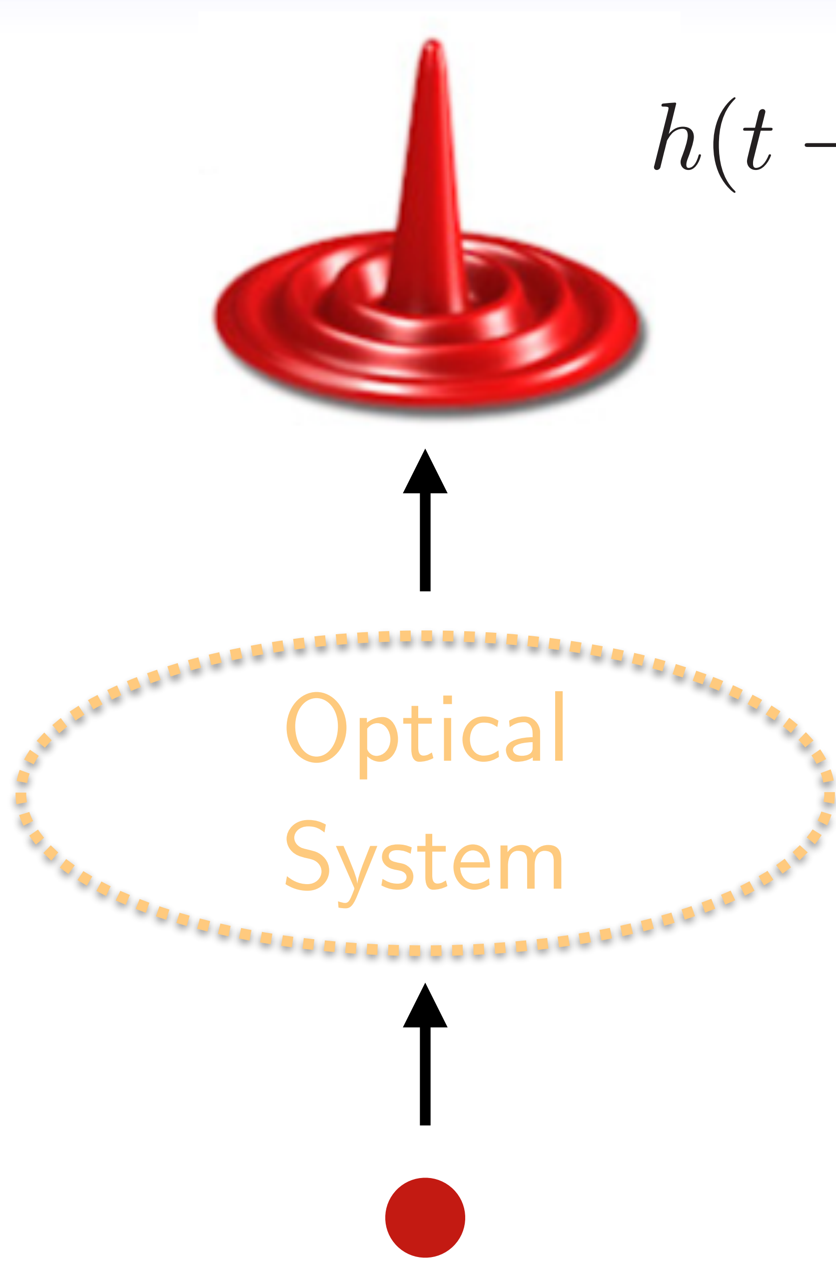
Super-Resolution Principle

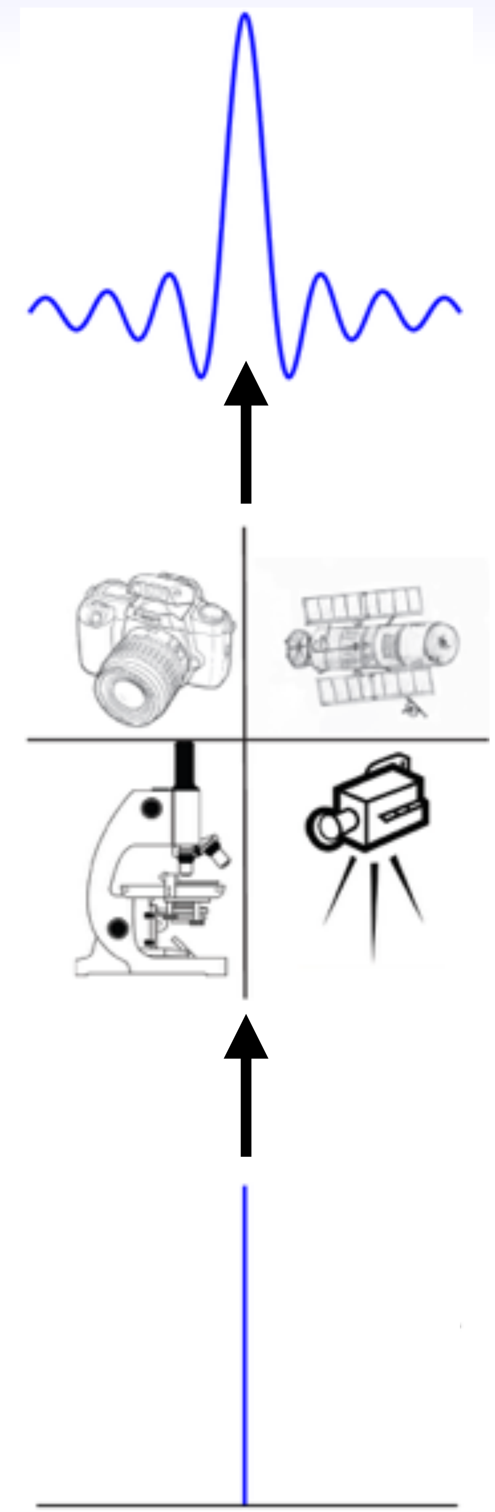
The resolving power of lenses, however perfect, is limited (Lord Rayleigh)





$h(t-\tau)$

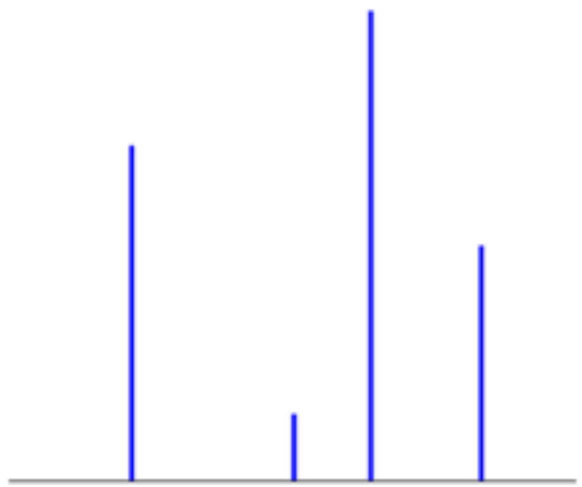
$\delta(t-\tau)$



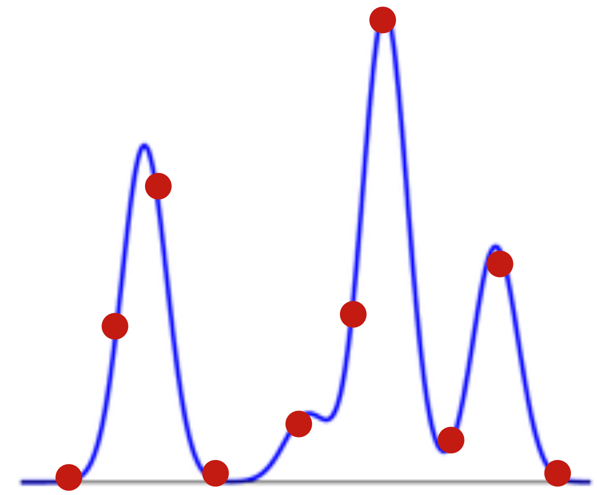
Super-Resolution Principle

Objective | Data

(spatial)

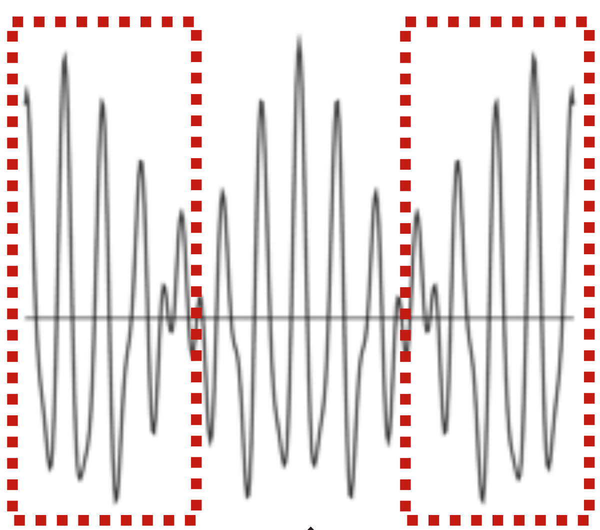


f

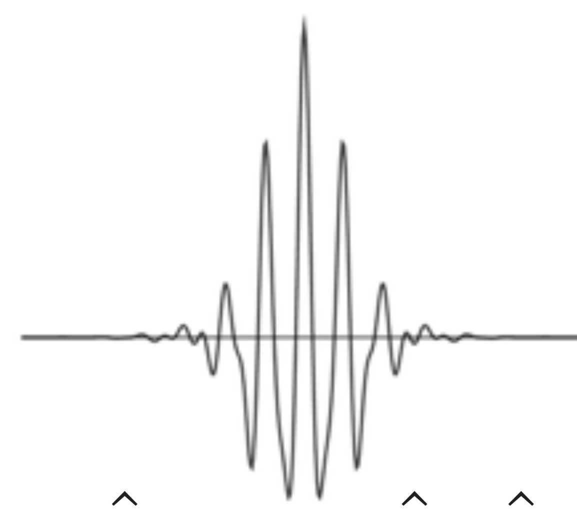


$f_{\text{obs}} = f * h$

(Fourier)



$\hat{f}$

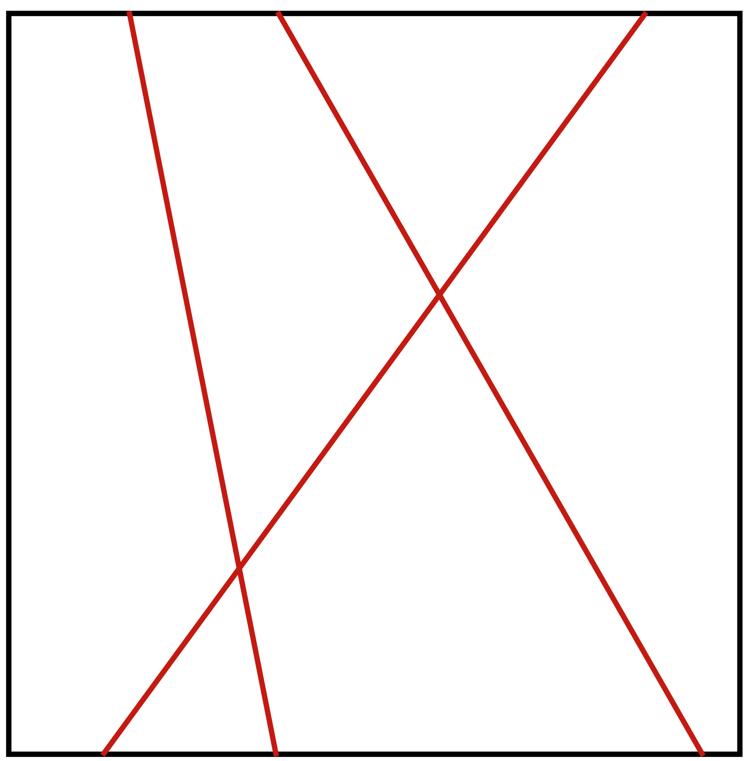


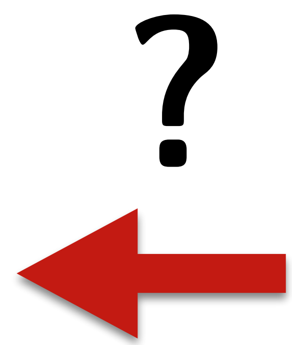
$\hat{f}_{\text{obs}} = \hat{f} \cdot \hat{h}$

h bandlimited

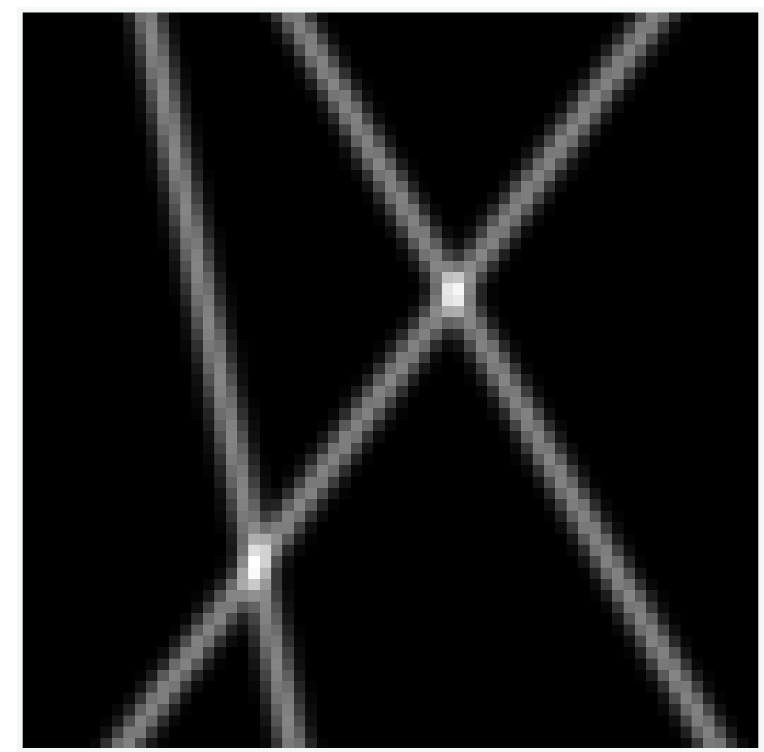
Super-Resolution Detection of Lines

Objective f

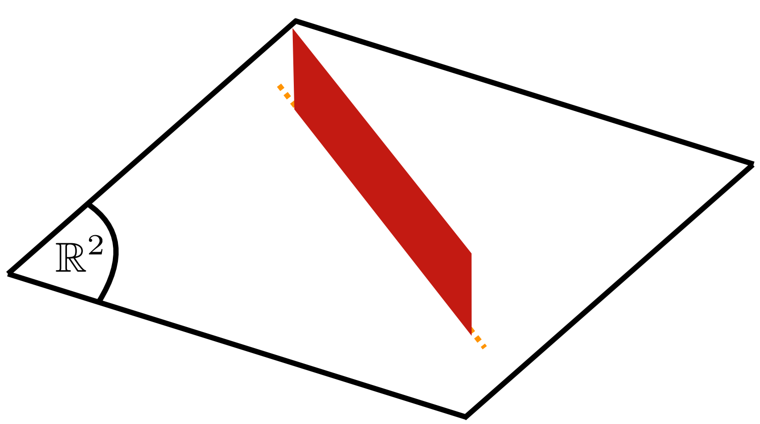




Data



$$f_{\text{obs}} = f * h$$



+ Recovering line parameters

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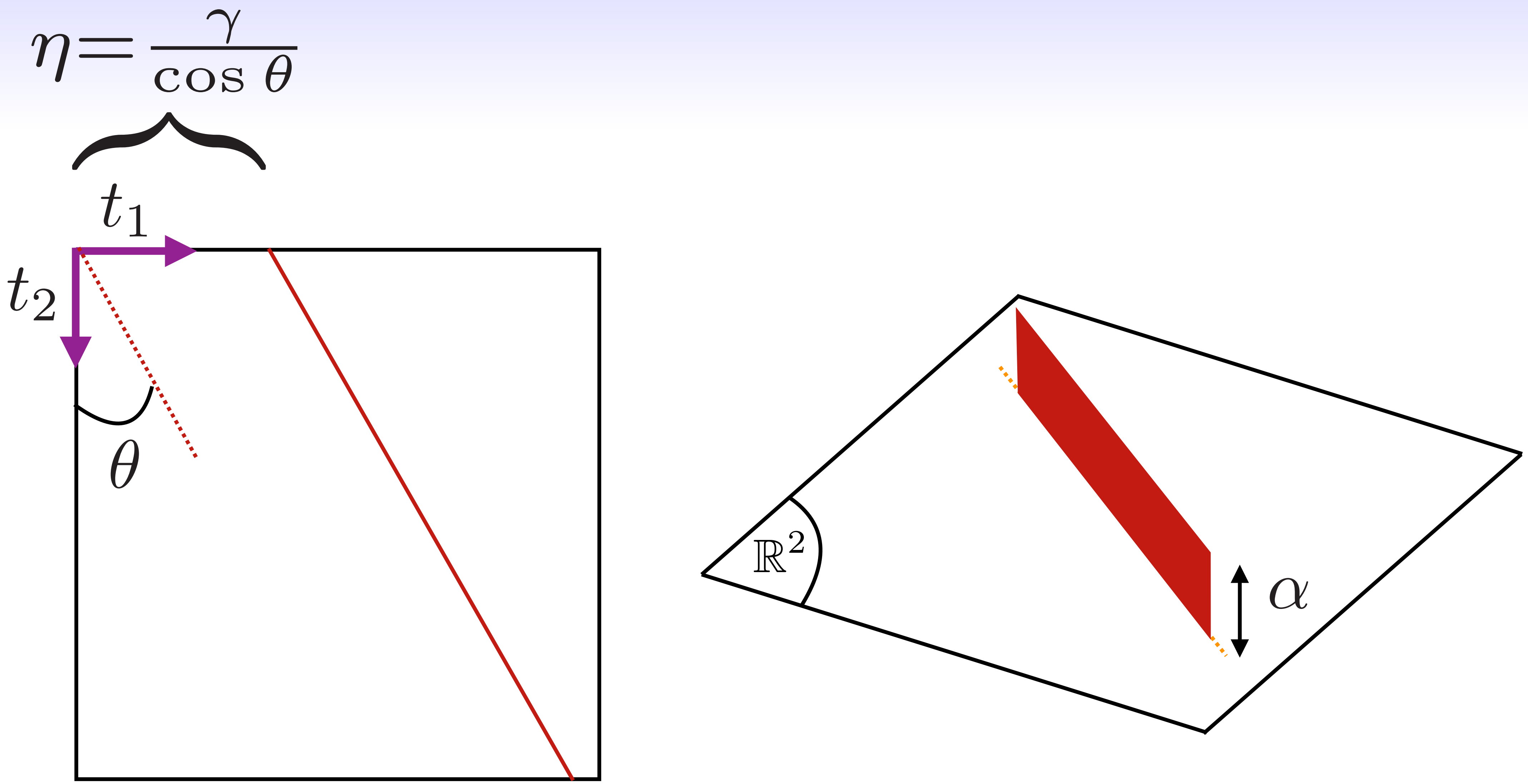
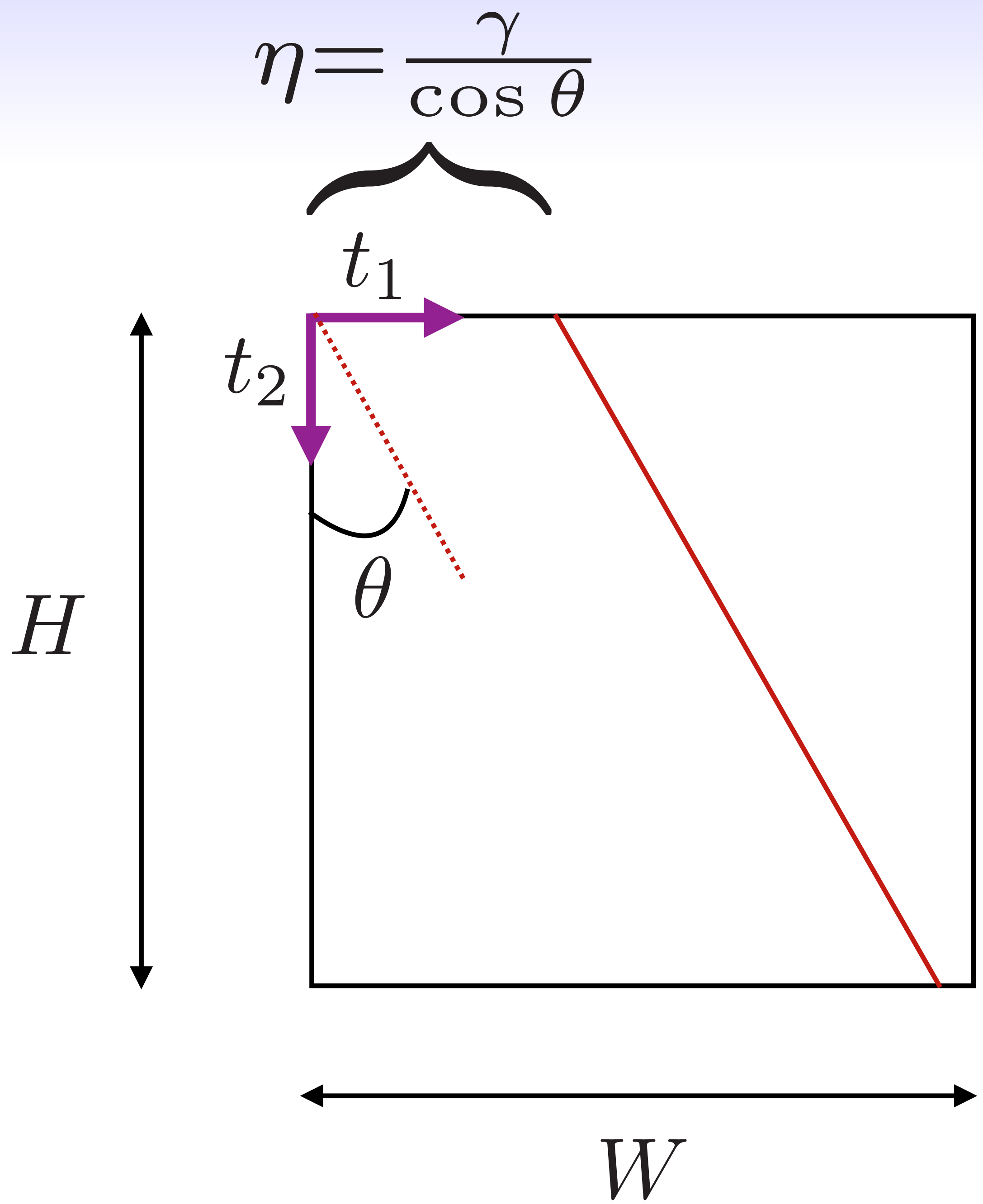
Problem formulation



$$s^\sharp : (t_1, t_2) \in \mathbb{P} \mapsto \alpha\delta\big(\cos(\theta)t_1 + \sin(\theta)t_2 - \gamma\big)$$

Problem formulation

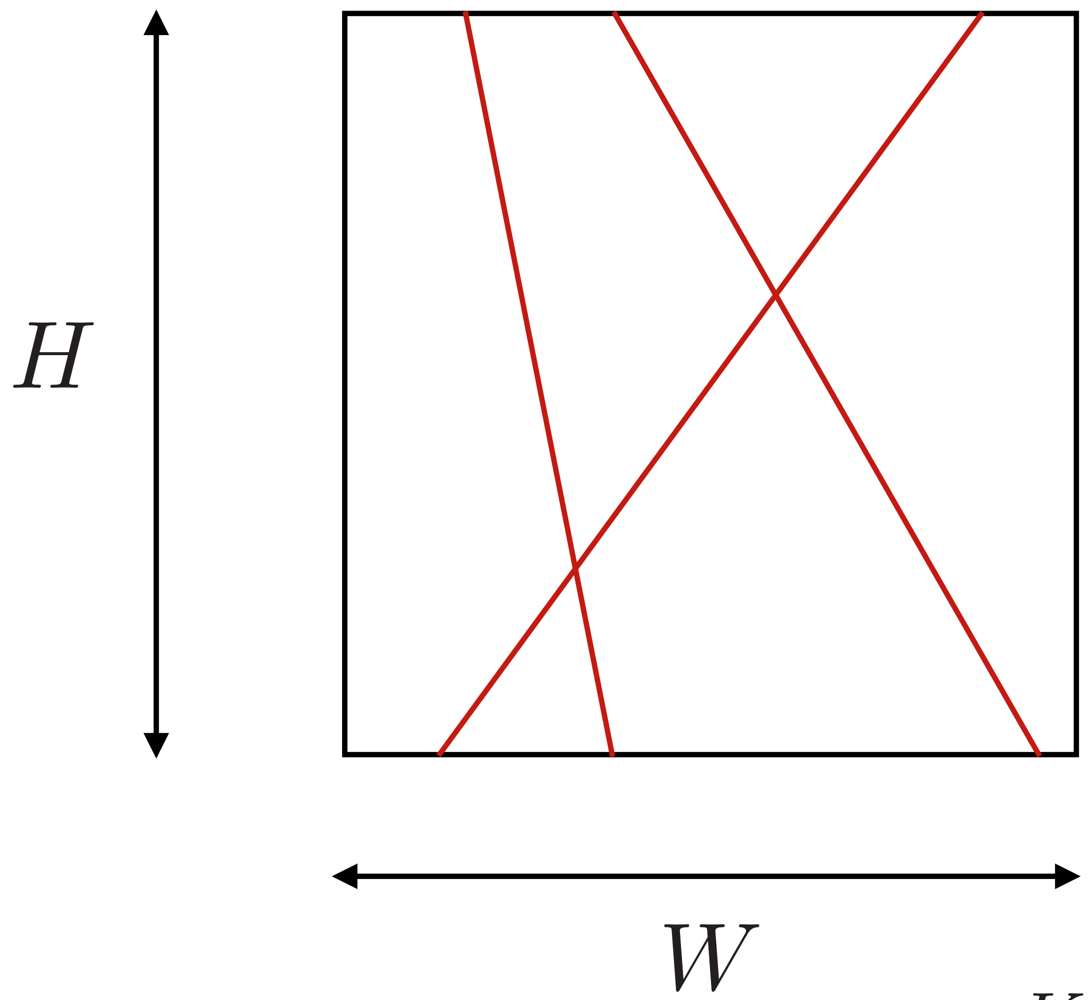


- Assumptions:
  - $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4}]$
  - $\mathbb{P} = \mathbb{R}/(W\mathbb{Z}) \times \mathbb{R}$
  - W, H odds

$$s^\sharp : (t_1, t_2) \in \mathbb{P} \mapsto \alpha\delta\big(\cos(\theta)t_1 + \sin(\theta)t_2 - \gamma\big)$$

Problem formulation

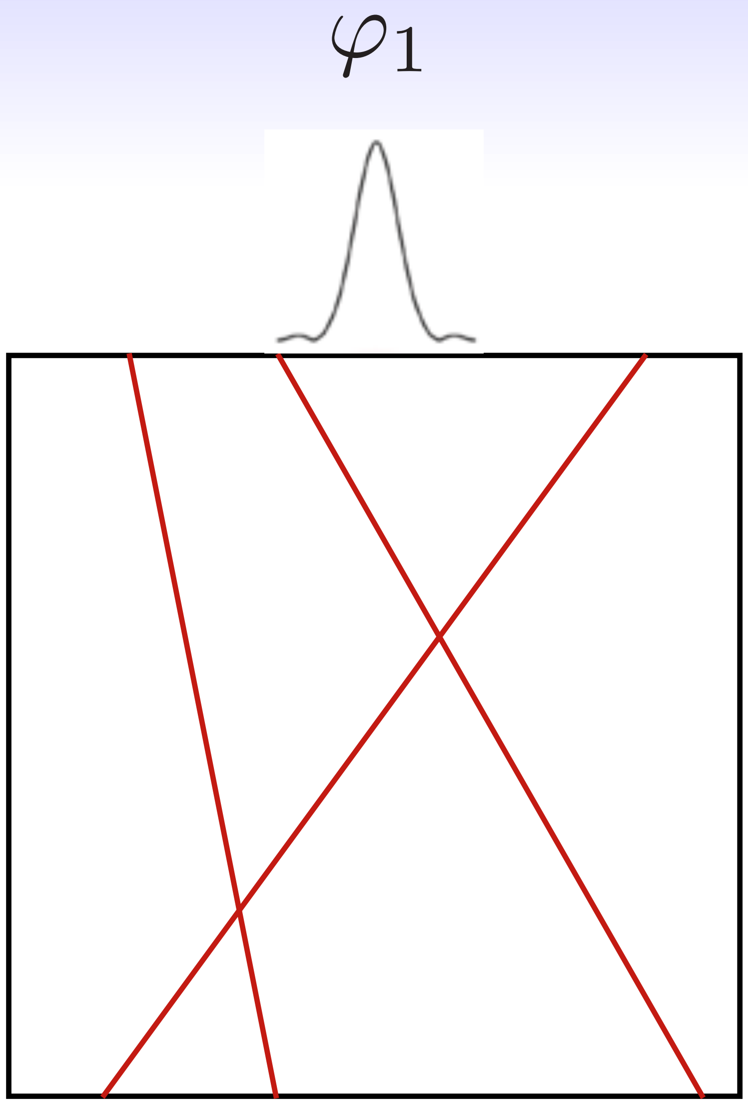
$$K = 3$$



- Assumptions:
  - $\theta_k \in (-\frac{\pi}{4}, \frac{\pi}{4}]$
  - $\mathbb{P} = \mathbb{R}/(W\mathbb{Z}) \times \mathbb{R}$
  - W, H odds

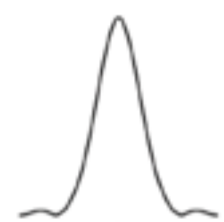
$$s^\sharp : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^{K} \alpha_k \delta\big(\cos(\theta_k)t_1 + \sin(\theta_k)t_2 - \gamma_k\big)$$

Problem formulation



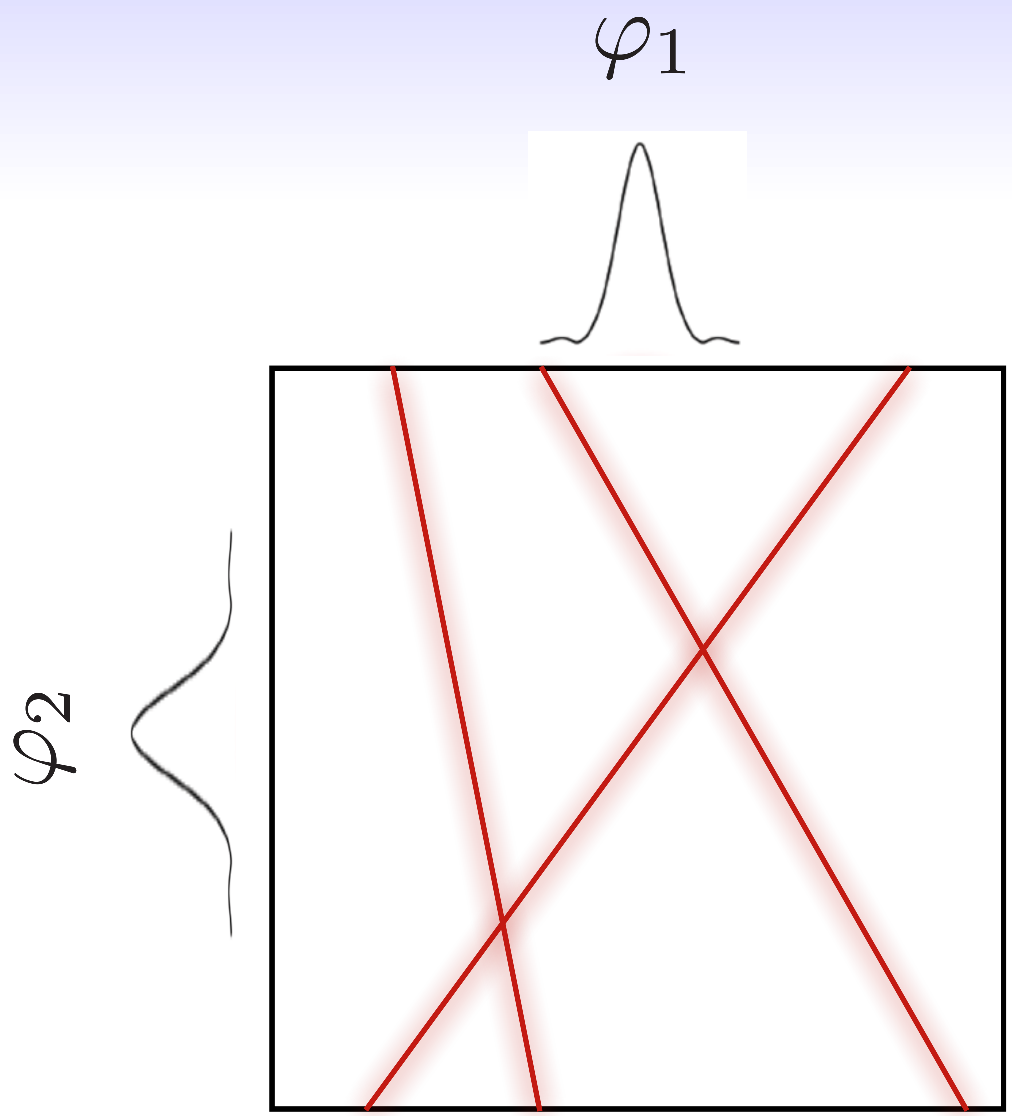
- Horizontal convolution

$$u^\sharp = s^\sharp * \phi_1$$



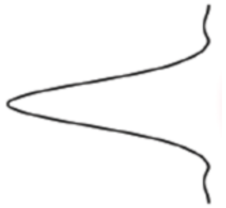
$$s^\sharp : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^{K} \alpha_k \delta\big(\cos(\theta_k) t_1 + \sin(\theta_k) t_2 - \gamma_k\big)$$

Problem formulation



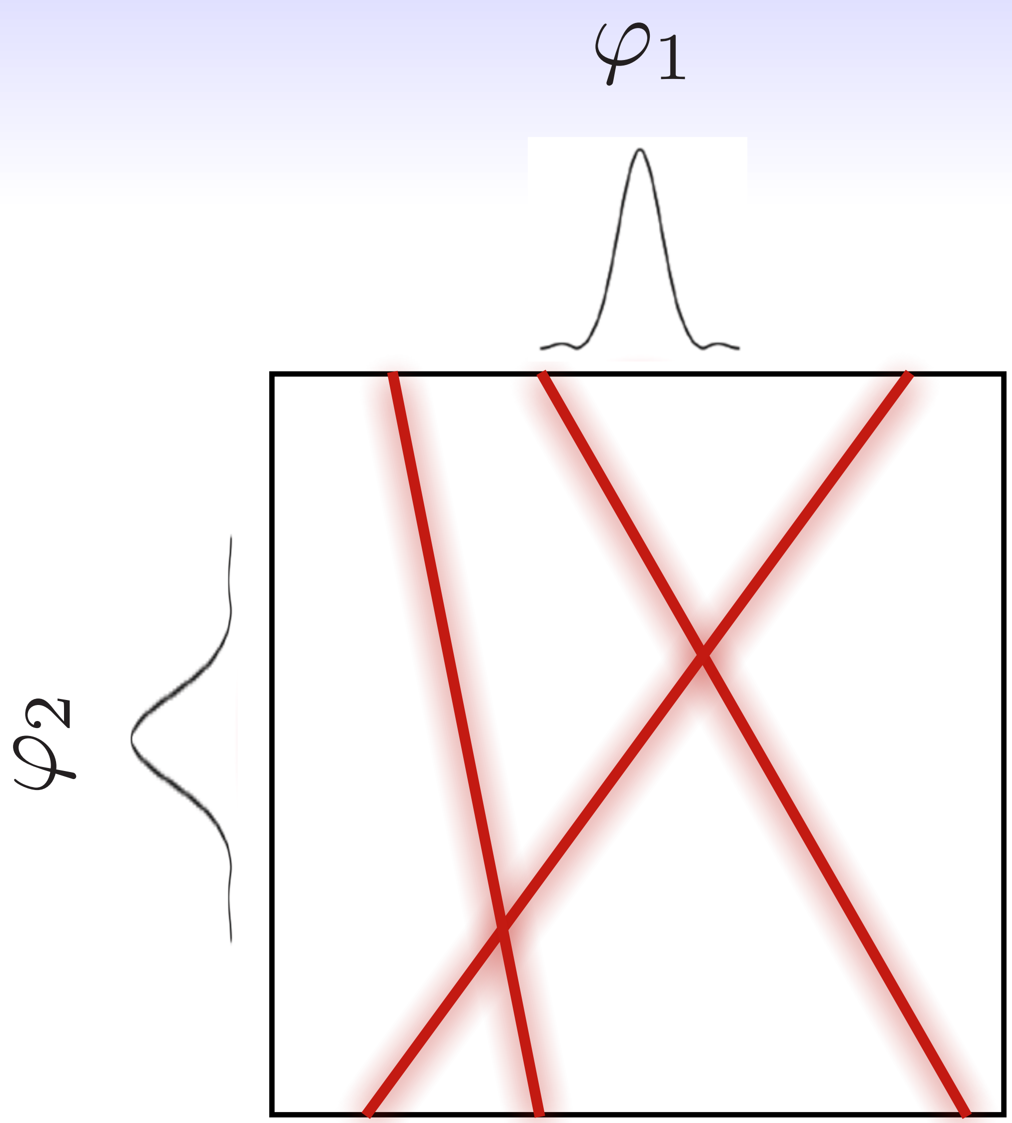
- Vertical convolution

$$s^\sharp * \phi = u^\sharp * \phi_2$$



$$u^\sharp : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^{K} \frac{\alpha_k}{\cos(\theta_k)} \varphi_1\Big(t_1 + \tan(\theta_k) t_2 + \frac{\gamma_k}{\cos(\theta_k)}\Big)$$

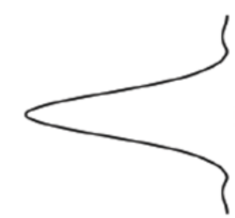
Problem formulation



$s^\sharp * \phi =$ 

- Vertical convolution

$$x^\sharp = s^\sharp * \phi = u^\sharp * \phi_2$$



$$\psi_k = \left(\frac{1}{\cos(\theta_k)}\varphi_1\left(\frac{\cdot}{\cos(\theta_k)}\right)\right) * \left(\frac{1}{\sin(\theta_k)}\varphi_2\left(\frac{\cdot}{\sin(\theta_k)}\right)\right)$$

$$s^\sharp * \phi : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^{K} \alpha_k \psi_k\left(\cos(\theta_k)t_1 + \sin(\theta_k)t_2 + \gamma_k\right)$$

Blur model

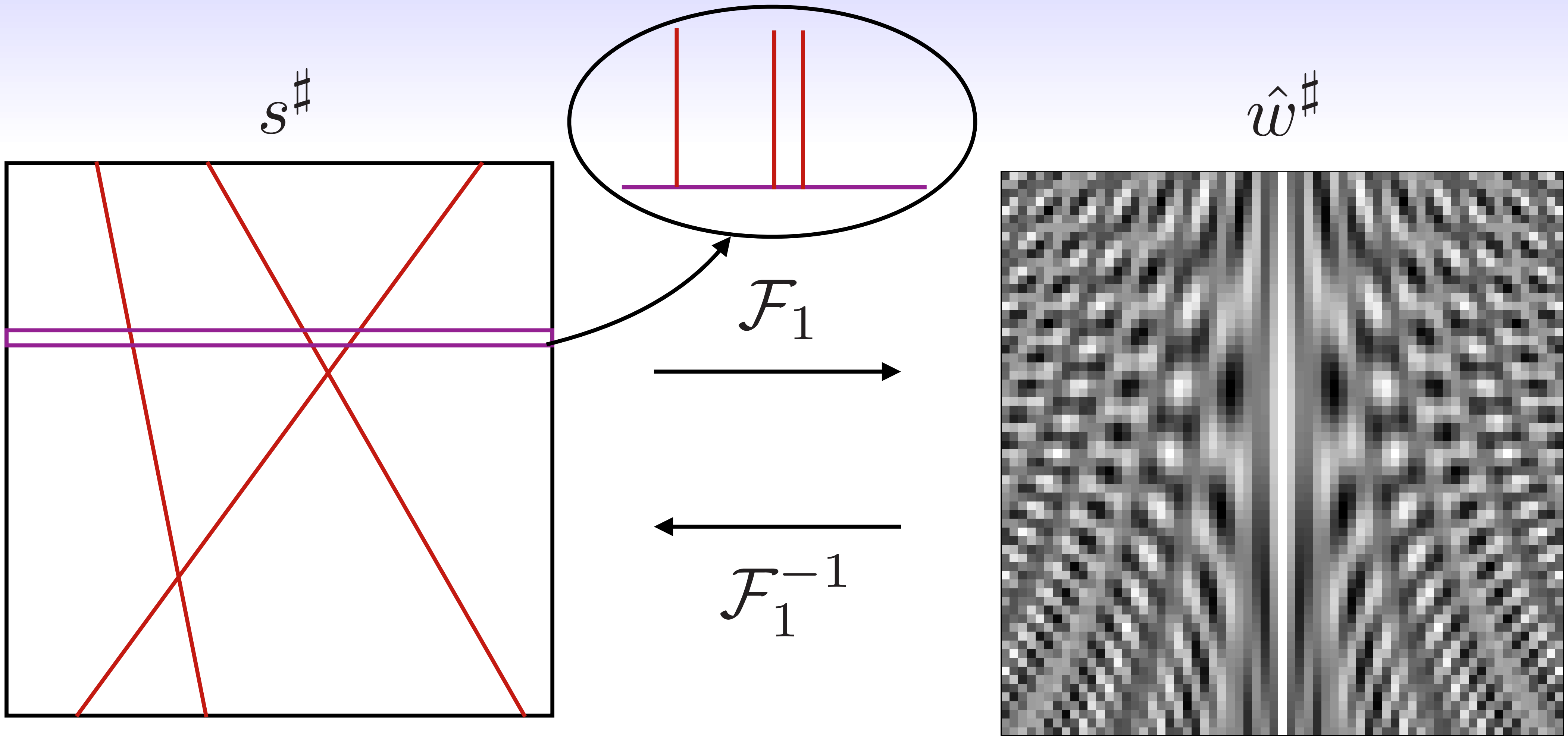
- Horizontal blur $\varphi_1 \in L^1([0, W))$
  - W − periodic
  - Bandlimited $c_m(\varphi_1) = 0$ for $|m| \geqslant (W+1)/2$
  - $\int_0^W \varphi_1 = 1$

Discrete filter

$g[n] = \varphi_1(n)$

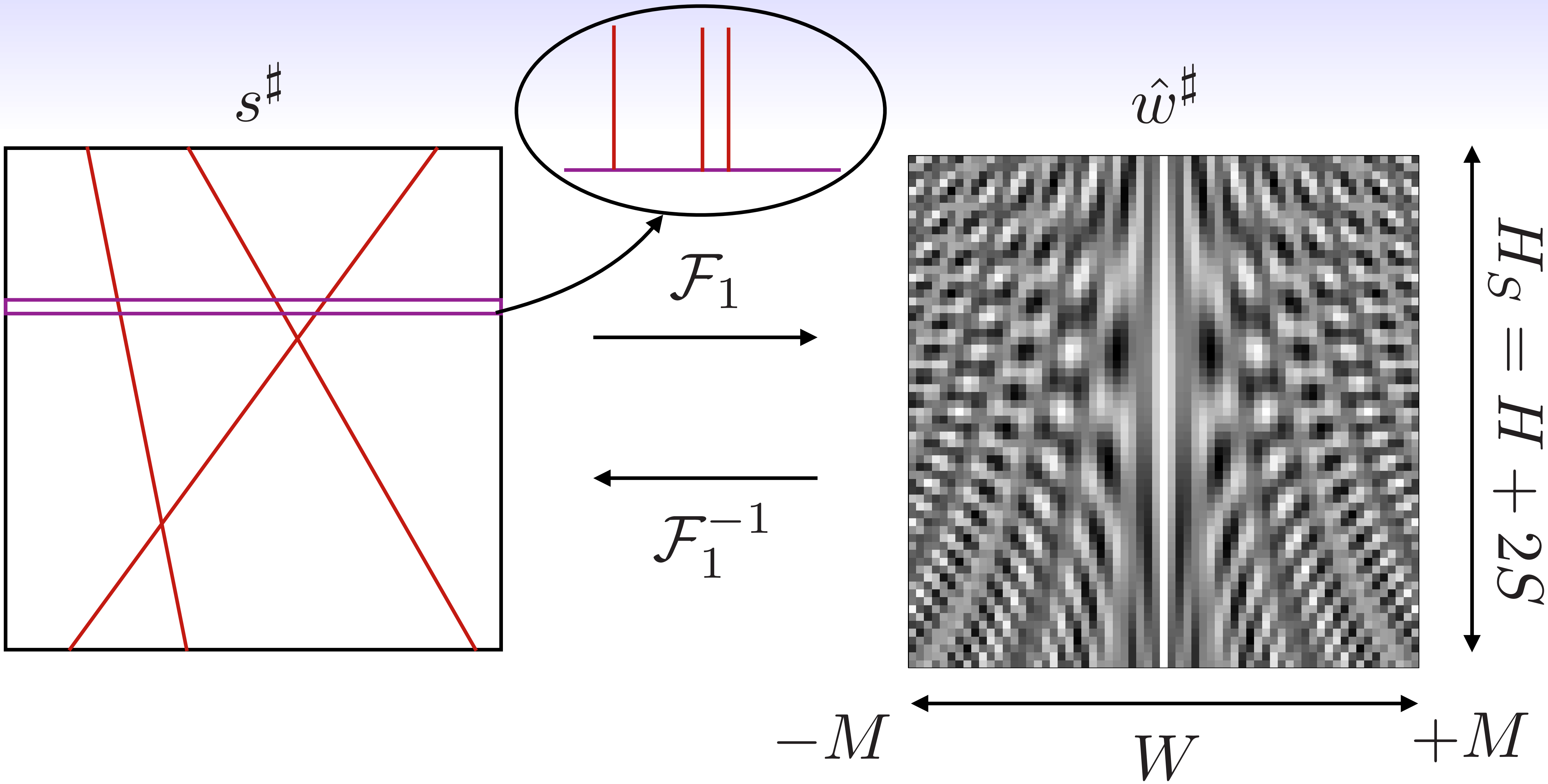
- Vertical blur $\varphi_2 \in L^1(\mathbb{R})$
  - Discrete filter $\big(h[n] = (\varphi_2 * \mathrm{sinc})(n)\big)_{n \in \mathbb{Z}}$ has compact support of length $2S+1$ for $S \in \mathbb{N}$
  - $\int_{\mathbb{R}} \varphi_2 = 1$

Spatial vs. Frequency domain



- $$s^\sharp(t_1, t_2) = \sum_{k=1}^{K} \alpha_k \delta\big(\cos(\theta_k)t_1 + \sin(\theta_k)t_2 - \gamma_k\big)$$
- $$\hat{w}^\sharp[m, n_2] = \sum_{k=1}^{K} \frac{\alpha_k}{\cos\theta_k} e^{j2\pi(\tan(\theta_k)n_2 - \eta_k)m/W}$$

Formulation as combinaison of atoms



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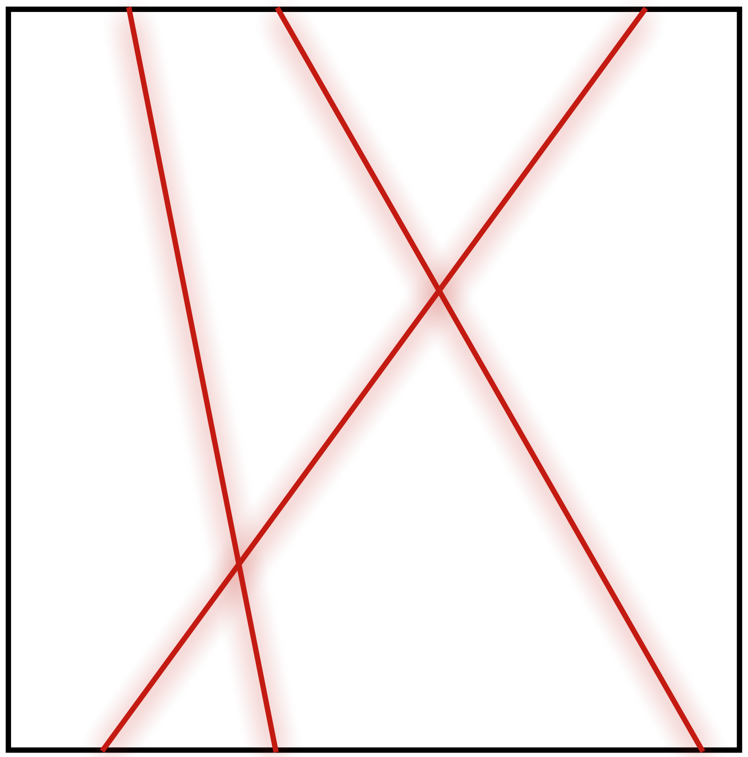
$$M = \frac{W-1}{2}$$

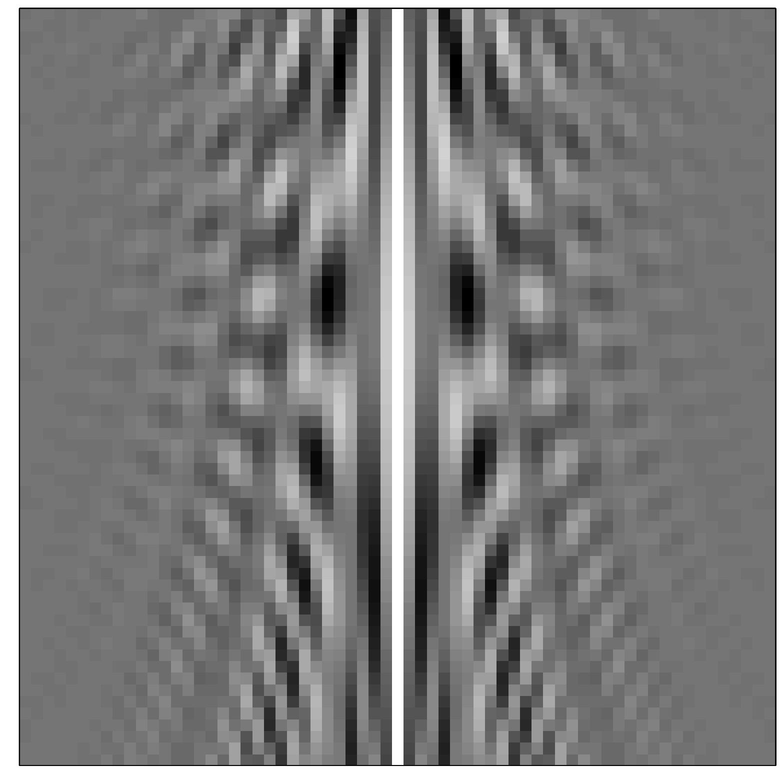
Spatial vs. Frequency domain

$v^\sharp$ $\qquad v^\sharp[n_1, n_2] = u^\sharp(n_1, n_2)$ $\qquad \hat{v}^\sharp$



$\xrightarrow{\mathcal{F}_1}$

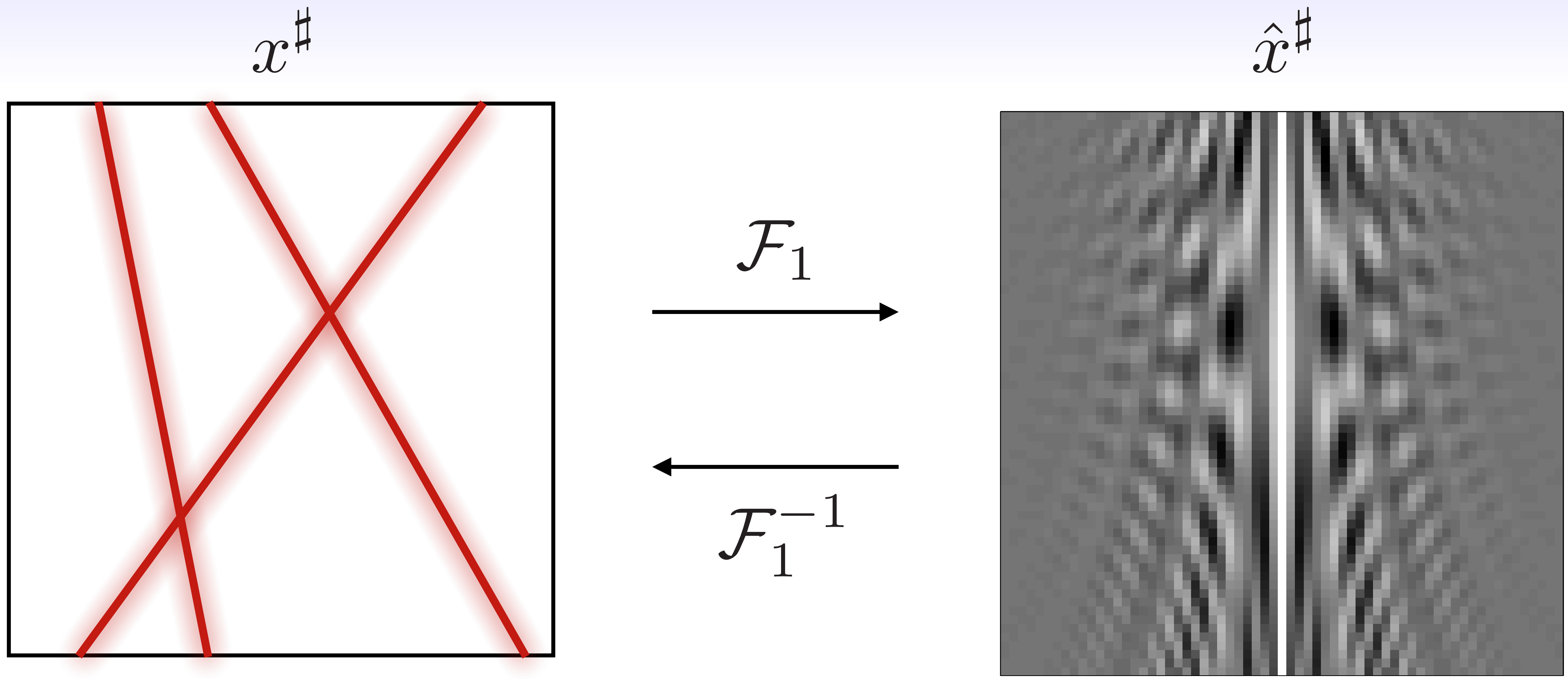
$\xleftarrow[\mathcal{F}_1^{-1}]{}$



- $$v^\sharp[n_1, n_2] = \sum_{k=1}^{K} \frac{\alpha_k}{\cos(\theta_k)} \varphi_1\Big(n_1 + \tan(\theta_k) n_2 - \eta_k\Big)$$

- $$\hat{v}^\sharp[m, n_2] = \hat{g}[m] \hat{w}^\sharp[m, n_2]$$

Spatial vs. Frequency domain



- $x^\sharp[n_1, n_2] = v^\sharp[n_1, :] * h$

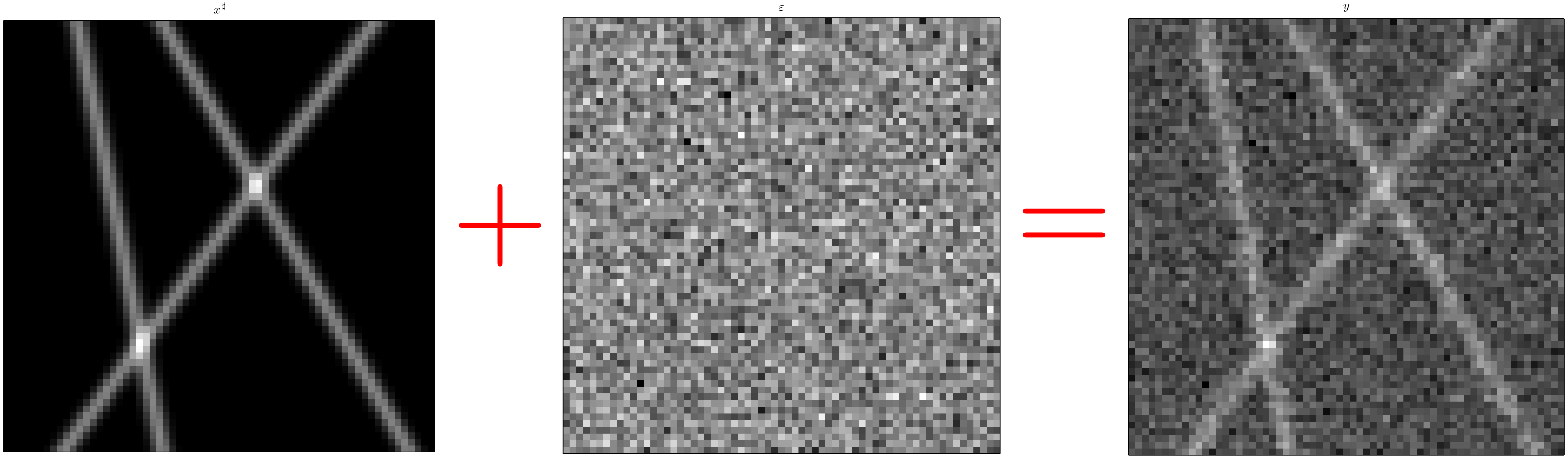
$$\mathbf{A}\hat{w}^\sharp = \hat{x}^\sharp$$

- $\hat{x}^\sharp[m, n_2] = \hat{v}^\sharp[m, :] * h = (\hat{g}[m]\hat{w}^\sharp[m, :]) * h$

Problem formulation

Data generation size: W, H blur: g, h filter size: S

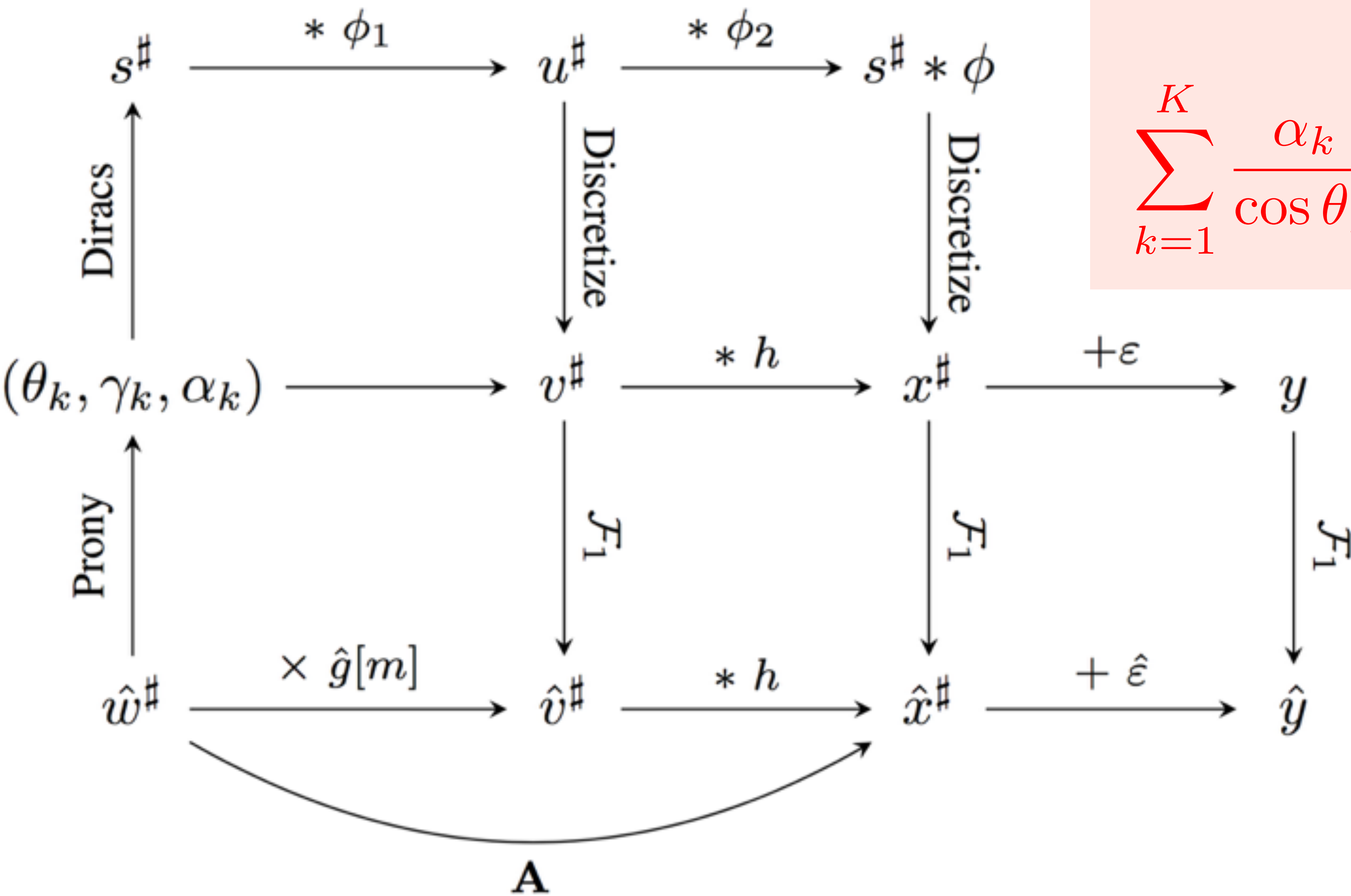
lines: $K, \{\theta_k, \alpha_k, \eta_k\}$ noise: $\varepsilon \sim \mathcal{N}(0, \zeta^2)$



$x^\sharp$ ε y

Inverse problem

Goal: From data $\hat{y}$ find the image of the exponentials $\hat{w}^\sharp$



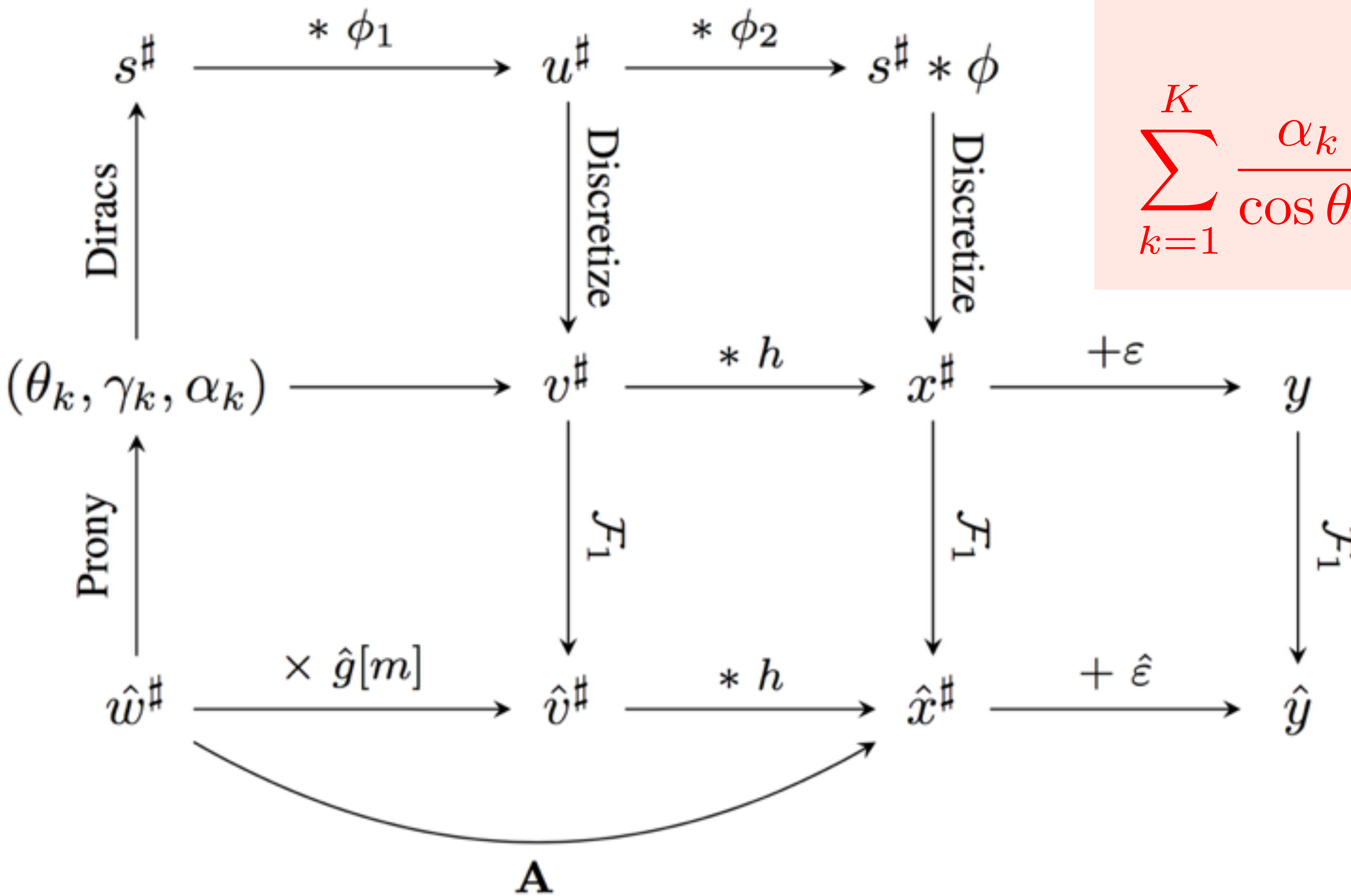
$$\hat{w}^\sharp[m, n_2] = \sum_{k=1}^{K} \frac{\alpha_k}{\cos\theta_k} e^{j2\pi(\tan(\theta_k)n_2 - \eta_k)\frac{m}{W}}$$

Inverse problem

$$\mathbf{A}\hat{w}^\sharp = \hat{x}^\sharp$$

Minimization problem

Goal: From data $\hat{y}$ find the image of the exponentials $\hat{w}^\sharp$



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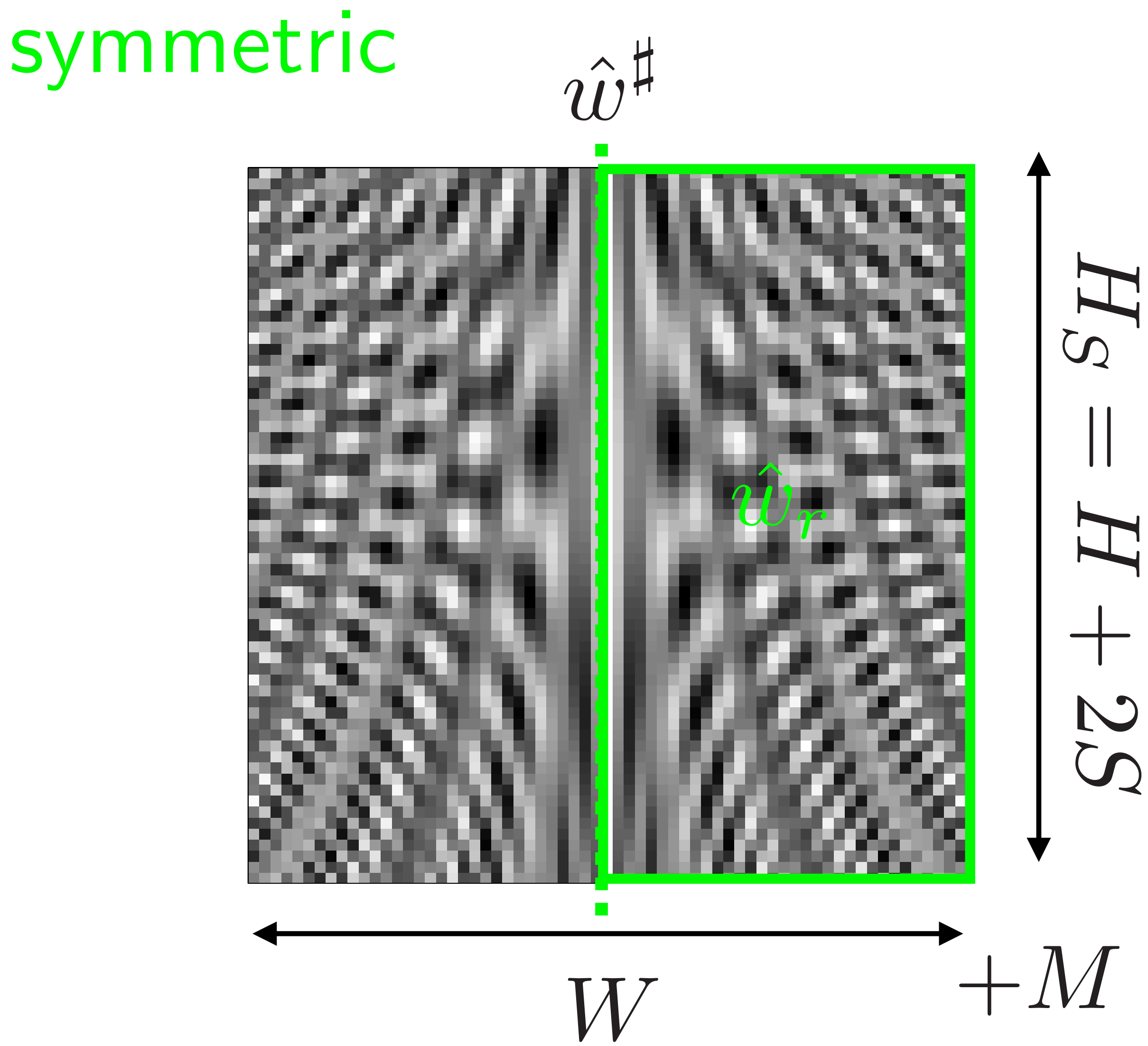
Minimization problem

$$\|\mathbf{A}\hat{w} - \hat{y}\|$$

Minimization problem

Goal: From data $\hat{y}$ find the image of the exponentials $\hat{w}^\sharp$



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Minimization problem

$$\|\mathbf{A}\hat{w}_r - \hat{y}_r\|$$

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Framework of atomic norm minimization

- **Simple model** from atomic set $\mathcal{A}$

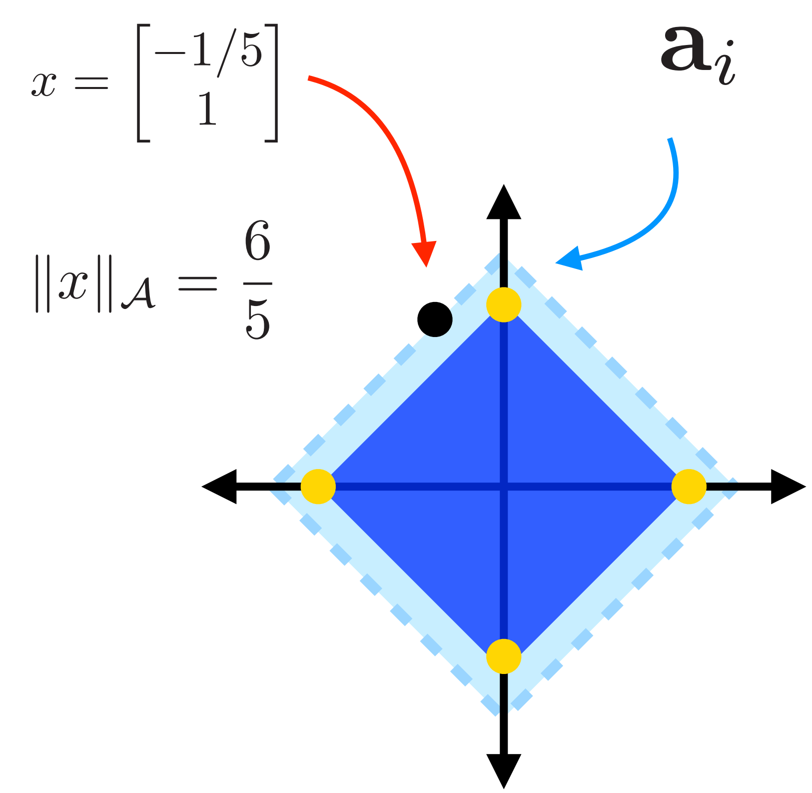
$$x = \sum_{i=1}^{r} c_i \mathbf{a}_i \qquad \mathbf{a}_i \in \mathcal{A}, \quad c_i \geqslant 0$$

model, rank, weights, atoms, dictionary

- $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$

- $\|x\|_{\mathcal{A}} = \inf \{ t > 0 : x \in t\operatorname{conv}(\mathcal{A}) \}$

- $\|x\|_{\mathcal{A}} = \inf\limits_{\substack{c_i \geqslant 0 \\ \mathbf{a}_i \in \mathcal{A}}} \left\{ \sum_{i=1}^{|\mathcal{A}|} c_i : x = \sum_{i=1}^{|\mathcal{A}|} c_i \mathbf{a}_i \right\}$



Framework of atomic norm minimization

- Sparse vectors $\mathcal{A} = \{\pm e_i\}_{i=1}^{N}$

  $\mathrm{conv}(\mathcal{A}) = \text{cross-polytope}$

  $\|x\|_{\mathcal{A}} = \|x\|_1$

- Low-rank matrices $\mathcal{A} = \{A : \mathrm{rank}(A) = 1, \|A\|_F = 1\}_{i=1}^{N}$

  $\mathrm{conv}(\mathcal{A}) = \text{nuclear norm ball}$

  $\|x\|_{\mathcal{A}} = \|x\|_*$

- Binary vectors $\mathcal{A} = \{\pm 1\}_{i=1}^{N}$

  $\mathrm{conv}(\mathcal{A}) = \text{hypercube}$

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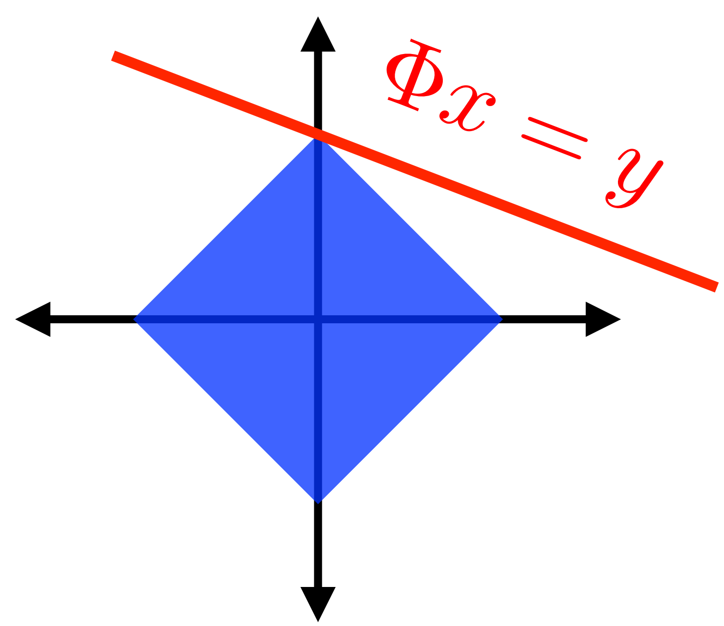
Minimize $\|x\|_{\mathcal{A}}$

Subject to $\Phi x = y$



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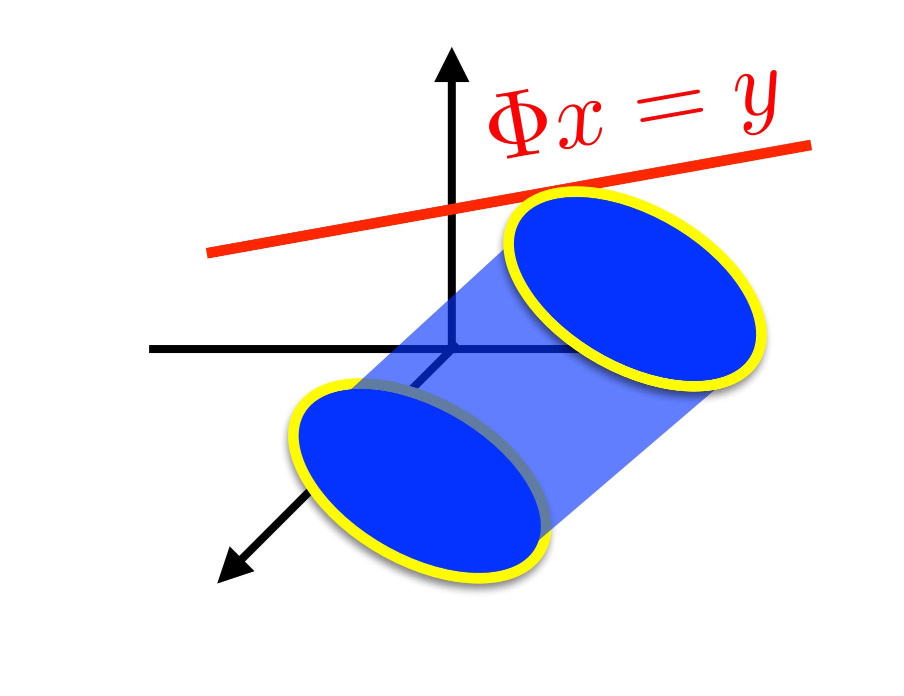

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Minimize $\|x\|_{\mathcal{A}}$

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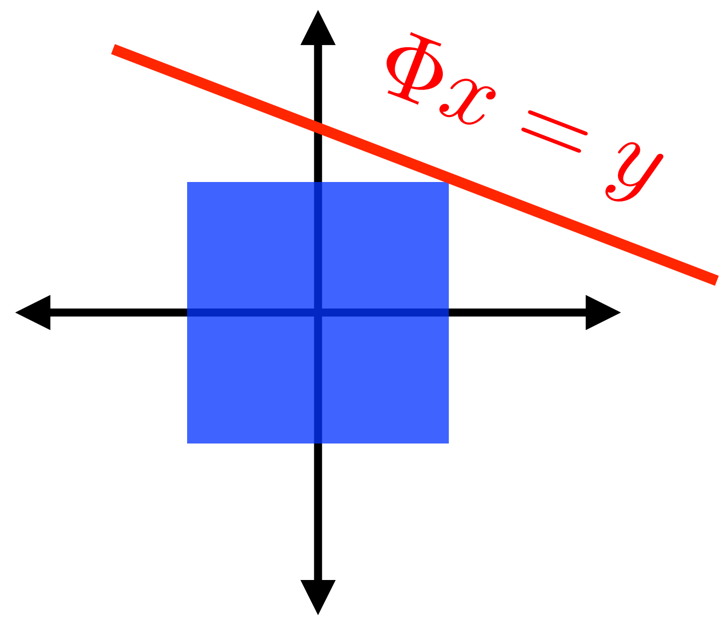
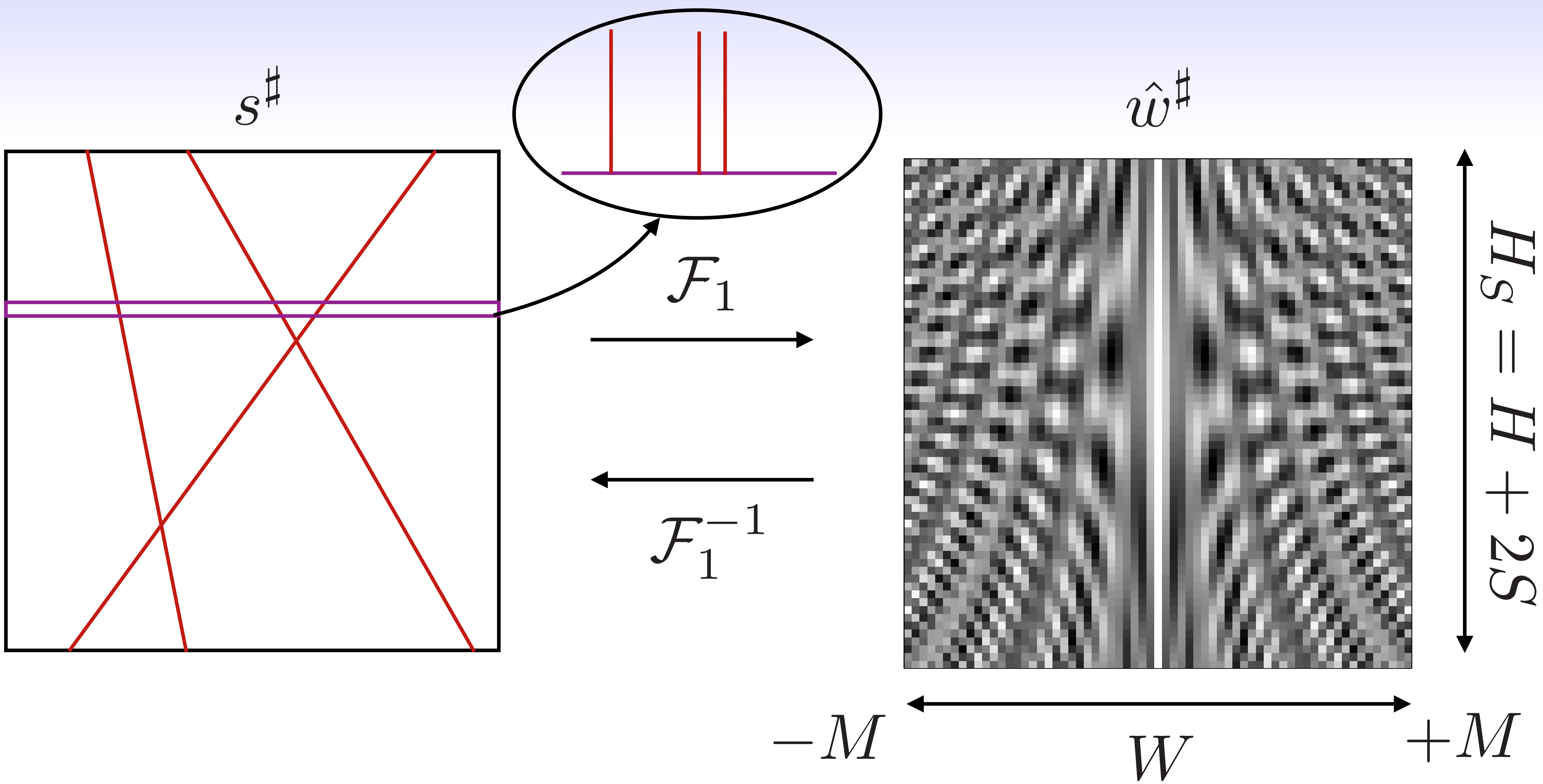


Formulation as combinaison of atoms



- $$\hat{w}^\sharp[m, n_2] = \sum_{k=1}^{K} \frac{\alpha_k}{\cos\theta_k} e^{j2\pi(\tan(\theta_k)n_2 - \eta_k)m/W}$$

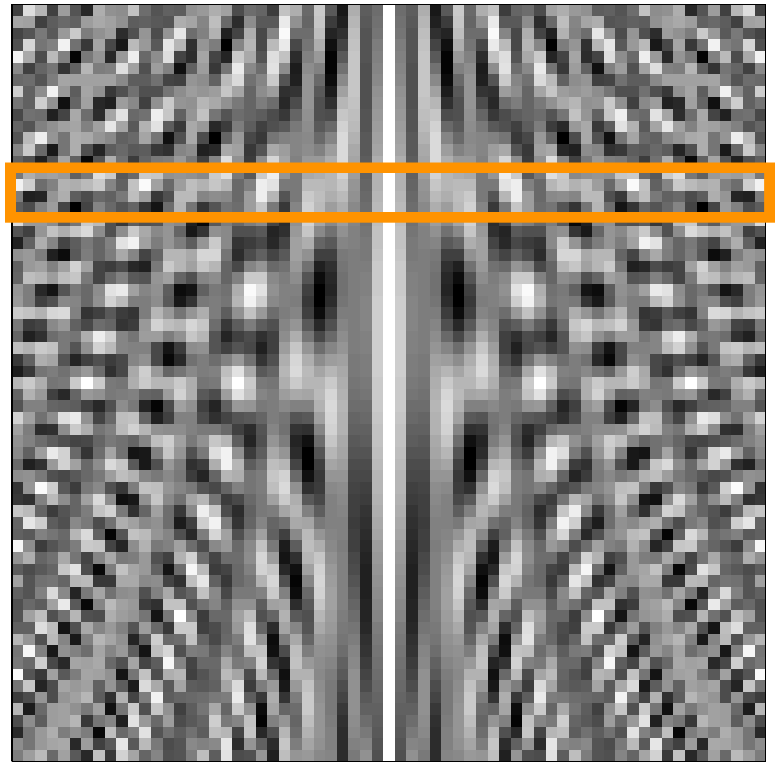
$$M = \frac{W-1}{2}$$

Formulation as combinaison of atoms

$$\mathcal{A} = \{a(f,\phi) \in \mathbb{C}^{|I|}, f \in [0,1], \phi \in [0, 2\pi)\}$$

- $[a(f,\phi)]_i = e^{j(2\pi f i + \phi)}, i \in I$
- $[a(f)]_i = e^{j2\pi f i}, i \in I$
- $I = \{-M, \dots, M\}$

$\hat{w}^\sharp$



$\leftarrow n_2$

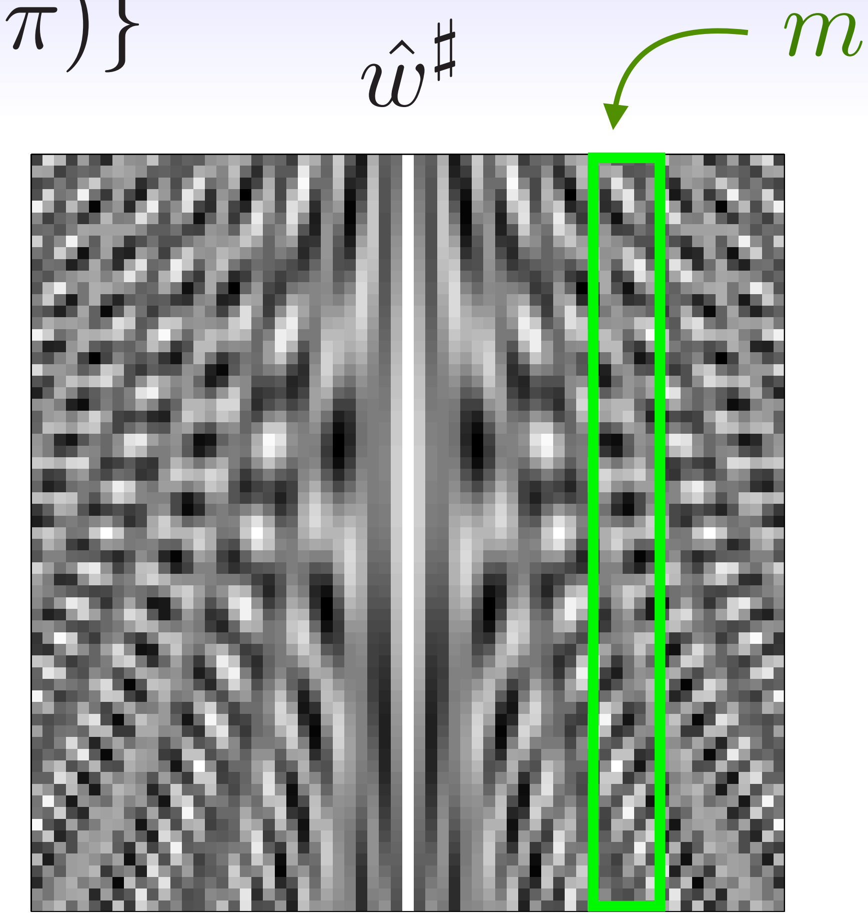
- $$\hat{w}^\sharp[m, n_2] = \sum_{k=1}^{K} \frac{\alpha_k}{\cos\theta_k} e^{j2\pi(\tan(\theta_k)n_2 - \eta_k)m/W}$$
- $$\hat{w}^\sharp_{n_2} = \hat{w}^\sharp[:, n_2] = \sum_{k=1}^{K} c_k a(f_{n_2,k})$$

$$c_k = \frac{\alpha_k}{\cos\theta_k}$$

$$f_{n_2,k} = \frac{\tan(\theta_k)n_2 - \eta_k}{W}$$

Formulation as combinaison of atoms

$$\mathcal{A} = \{a(f,\phi) \in \mathbb{C}^{|I|}, f \in [0,1], \phi \in [0,2\pi)\}$$



- $[a(f,\phi)]_i = e^{j(2\pi f i+\phi)}, i \in I$
- $[a(f)]_i = e^{j2\pi f i}, i \in I$
- $I = \{0,\ldots,H_S-1\}$

- $\hat{w}^\sharp[m,n_2] = \sum_{k=1}^{K} \frac{\alpha_k}{\cos\theta_k} e^{j2\pi(\tan(\theta_k)n_2-\eta_k)m/W}$
- $\hat{w}^\sharp_m = \hat{w}^\sharp[m,:] = \sum_{k=1}^{K} c_k a(f_{m,k},\phi_{m,k})^T$

$$c_k = \frac{\alpha_k}{\cos\theta_k}$$

$$f_{m,k} = \frac{\tan(\theta_k)m}{W}$$

$$\phi_{m,k} = -\frac{2\pi\eta_k m}{W}$$

Formulation as combinaison of atoms

$$\|x\|_{\mathcal{A}} = \inf_{\substack{c'_k \geqslant 0 \\ f'_k \in [0,1] \\ \phi'_k \in [0,2\pi)}} \left\{ \sum_k c'_k : x = \sum_k c'_k a(f'_k, \phi'_k) \right\}.$$

- $\hat{w}^\sharp_m = \hat{w}^\sharp[m,:] = \sum_{k=1}^{K} c_k a(f_{m,k}, \phi_{m,k})^T$
- $\hat{w}^\sharp_{n_2} = \hat{w}^\sharp[:,n_2] = \sum_{k=1}^{K} c_k a(f_{n_2,k})$

$\|\hat{w}^\sharp_m\|_{\mathcal{A}}$? $\qquad$ $\|\hat{w}^\sharp_{n_2}\|_{\mathcal{A}}$?

Caratheodory theorem

Toeplitz Operator:

$$\mathbf{T} : (x_1, \dots, x_N) \mapsto \begin{pmatrix} x_1 & x_2^* & \cdots & x_N^* \\ x_2 & x_1 & \cdots & x_{N-1}^* \\ \vdots & \vdots & \ddots & \vdots \\ x_N & x_{N-1} & \cdots & x_1 \end{pmatrix}$$

- **Caratheodory theorem.** A vector x of length $N = 2M + 1$ is a positive combinaison $(c_k > 0)$ of $K \leqslant M + 1$ atoms $a(f_k)$ if and only if $\mathbf{T}(x) \succcurlyeq 0$.
  Also this decomposition is unique if $K \leqslant M$.

Atomic norm of the rows

- $\hat{w}^{\sharp}_{n_2} = \hat{w}^{\sharp}[:, n_2] = \sum_{k=1}^{K} c_k a(f_{n_2,k})$

$$\|\hat{w}^{\sharp}_{n2}\|_{\mathcal{A}} = \inf \left\{ \sum_k c'_k : \hat{w}^{\sharp}_{n2} = \sum_k c'_k a(f'_k, \phi'_k) \right\} = \sum_{k=1}^{K} c_k$$

- **Caratheodory theorem.** A vector x of length $N = 2M + 1$ is a positive combinaison $(c_k > 0)$ of $K \leqslant M + 1$ atoms $a(f_k)$ if and only if $\mathbf{T}(x) \succcurlyeq 0$.
  Also this decomposition is **unique** if $K \leqslant M$.

Atomic norm of the columns

- $\hat{w}^\sharp_m = \hat{w}^\sharp[m,:] = \sum_{k=1}^{K} c_k a(f_{m,k}, \phi_{m,k})^T$ $\quad c_k e^{j\phi_{m,k}} \in \mathbb{C}$

$$\|\hat{w}^\sharp_m\|_{\mathcal{A}} = \inf\left\{\sum_k c'_k : \hat{w}^\sharp_m = \sum_k c'_k a(f'_k, \phi'_k)\right\} \leqslant \sum_{k=1}^{K} c_k$$

- **Caratheodory theorem.** A vector x of length $N = 2M+1$ is a **positive combinaison** $(c_k > 0)$ of $K \leqslant M+1$ atoms $a(f_k)$ if and only if $\mathbf{T}(x) \succcurlyeq 0$. Also this decomposition is unique if $K \leqslant M$.

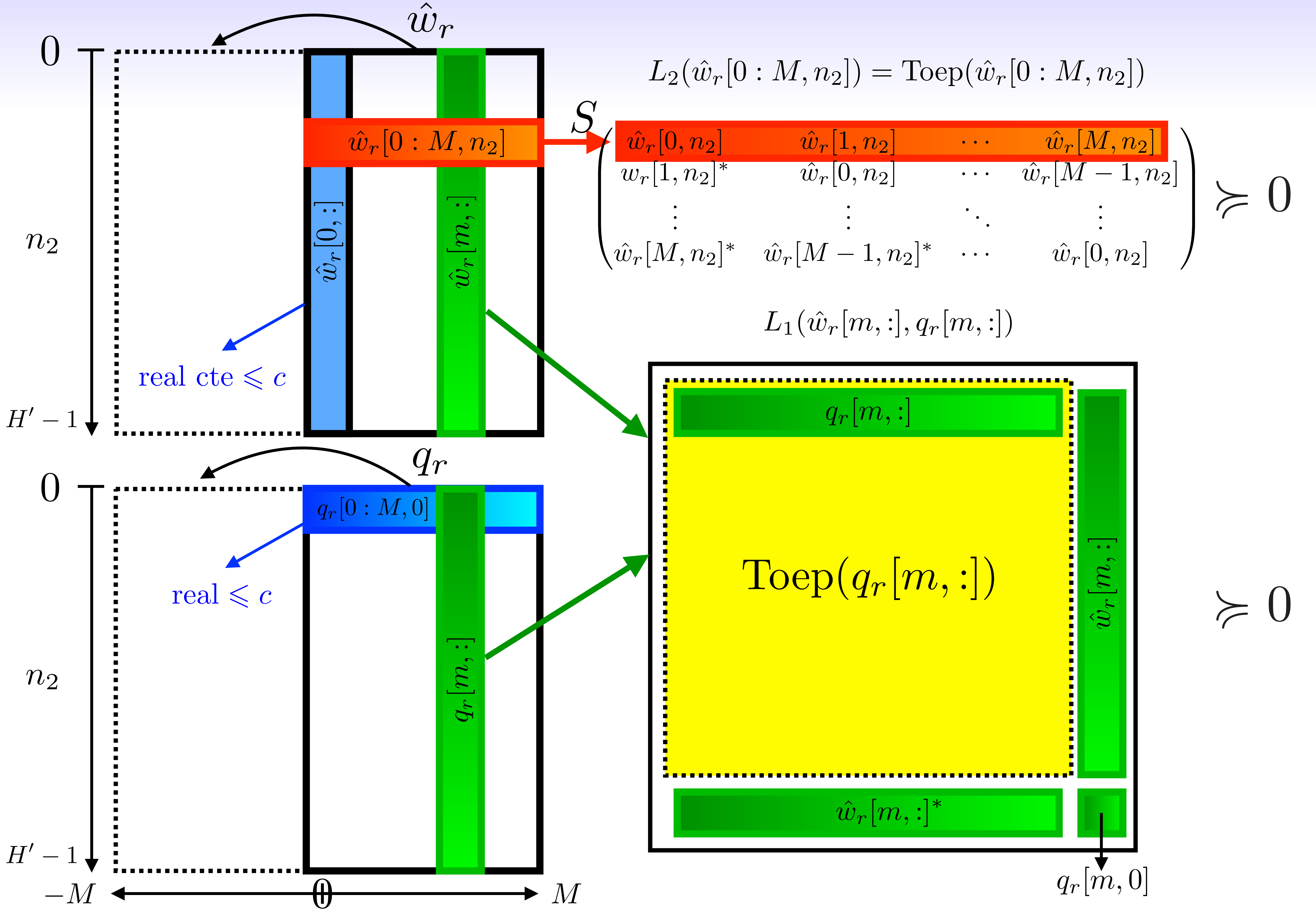
Atomic norm via a semidefinite program

Proposition. The atomic norm $\|\hat{w}_m^\sharp\|_\mathcal{A}$ can be characterized by the following semidefinite program:

$$\|\hat{w}_m^\sharp\|_\mathcal{A} = \min_{q_m \in \mathbb{C}^{H_S}} \left\{ q_m[0] : \begin{bmatrix} \mathbf{T}(q_m) & \hat{w}_m^\sharp \\ (\hat{w}_m^\sharp)^* & q_m[0] \end{bmatrix} \succcurlyeq 0 \right\}.$$

improvement of [Bhaskar et al.,2013]

Constraints



Minimization problem

$$\hat{w}_r^\sharp \in \arg\min_{\hat{w}_r, q_r} \frac{1}{2} \| A\hat{w}_r - \hat{y}_r \|^2$$

Subject to

$$\begin{cases} \forall n_2 = 0, \ldots, H_S - 1, \forall m = 0, \ldots, M, \\ \hat{w}_r[0, n_2] = \hat{w}_r[0, 0] \leqslant c, \\ q_r[m, 0] \leqslant c, \\ L_1(\hat{w}_r[m, :], q_r[m, :]) \succcurlyeq 0, \\ L_2(\hat{w}_r[:, n_2]) \succcurlyeq 0 \end{cases}$$

Minimization problem

- $X = (\hat{w}_r, q_r) \in \mathcal{H}$

$$\hat{w}_r^\sharp \in \underset{\hat{w}_r, q_r}{\arg\min} \; \overbrace{\frac{1}{2}\|A\hat{w}_r - \hat{y}_r\|^2}^{G(X)}$$

Subject to

$$\begin{cases} \forall n_2 = 0, \ldots, H_S - 1, \forall m = 0, \ldots, M, \\ \hat{w}_r[0, n_2] = \hat{w}_r[0,0] \leqslant c, \\ q_r[m, 0] \leqslant c, \\ L_1(\hat{w}_r[m,:], q_r[m,:]) \succcurlyeq 0, \\ L_2(\hat{w}_r[:, n_2]) \succcurlyeq 0 \end{cases}$$

boundary constraints $\mathcal{B}$ (lines 2–3)

cone of positive matrices $\mathcal{C}$ (lines 4–5)

- $L_1^m : (\hat{w}_r, q_r) \mapsto L_1(\hat{w}_r[m,:], q_r[m,:])$
- $L_2^{n_2} : (\hat{w}_r, q_r) \mapsto L_2(\hat{w}_r[:, n_2])$

Minimization problem

$$X^\sharp = \underset{X=(\hat{w}_r, q_r)\in\mathcal{H}}{\arg\min} \left\{ \frac{1}{2}\|\mathbf{A}\hat{w}_r - \hat{y}_r\|^2_{\mathcal{W}} + \iota_{\mathcal{B}}(X) + \sum_{m=0}^{M} \iota_{\mathcal{C}}(L_1^m(X)) + \sum_{n_2=0}^{H_S-1} \iota_{\mathcal{C}}(L_2^{n_2}(X)) \right\}$$

$$X^\star = \underset{X\in\mathcal{H}}{\arg\min} \{G(X) + \mathbf{H}(\mathbf{L}(X))\}$$

Chambole-Pock

- $G = \frac{1}{2}\|\mathbf{A}\cdot - \hat{y}_r\|^2_{\mathcal{W}}$
- $\mathbf{H}x = \sum_{i=0}^{N} H_i x_i$
- $\mathbf{L}x = (L_1, \ldots, L_N)$

- $L_i \in \{L_1^m, L_2^{n_2}\} \quad H_i = \iota_{\mathcal{C}} \quad i < N$
- $L_N = \mathrm{Id} \quad H_N = \iota_{\mathcal{B}} \quad i = N$

$$N = M + H_S + 1$$

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Primal-dual algorithm

Proximal operator

[Condat ,2013]

Conditions
- $\tau\sigma \|\mathbf{L}\|^2 = 1$
- $\forall n \in \mathbb{N}, \rho_n \in]0,2[$
- $\sum_{n\in\mathbb{N}} \rho_n(2-\rho_n) = +\infty$

Primal variables
- $\tilde{x}_{n+1} = \operatorname{prox}_{\tau G}\left(x_n - \tau \sum_{i=0}^{N} L_i^* z_{i,n}\right)$
- $x_{n+1} = \rho_n \tilde{x}_{n+1} + (1-\rho_n)x_n$

Dual variables
- $\forall i = 0, \ldots, N$
  - $\tilde{z}_{i,n+1} = \operatorname{prox}_{\sigma H_i^*}\left(z_{i,n} + \sigma L_i(2\tilde{x}_{n+1} - x_n)\right)$
  - $z_{i,n+1} = \rho_n \tilde{z}_{i,n+1} + (1-\rho_n) z_{i,n}$

converge toward a solution $(x^\star, z_0^\star, \ldots, z_N^\star)$ of the problem.

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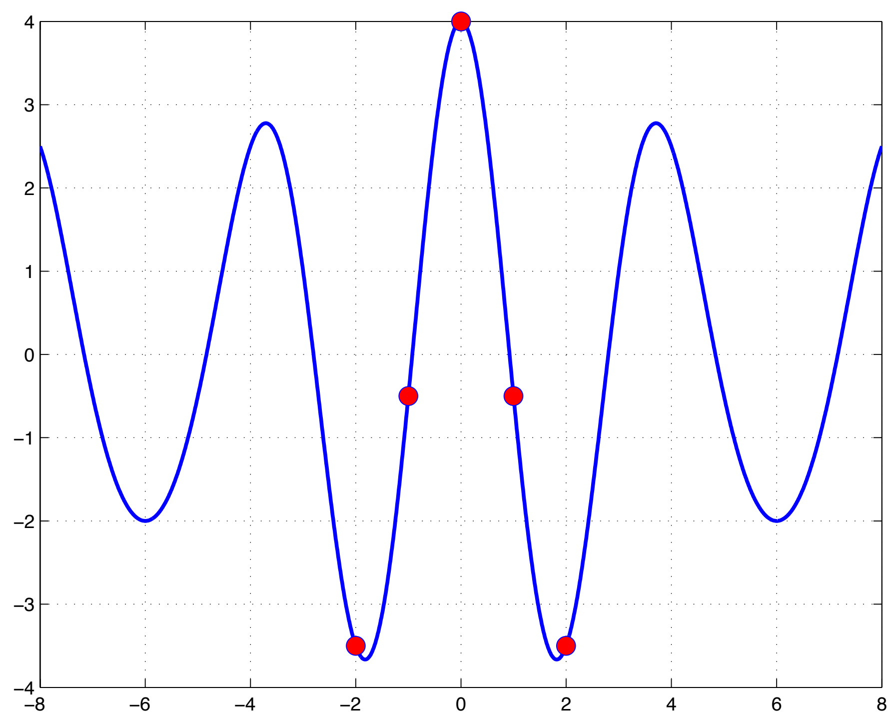
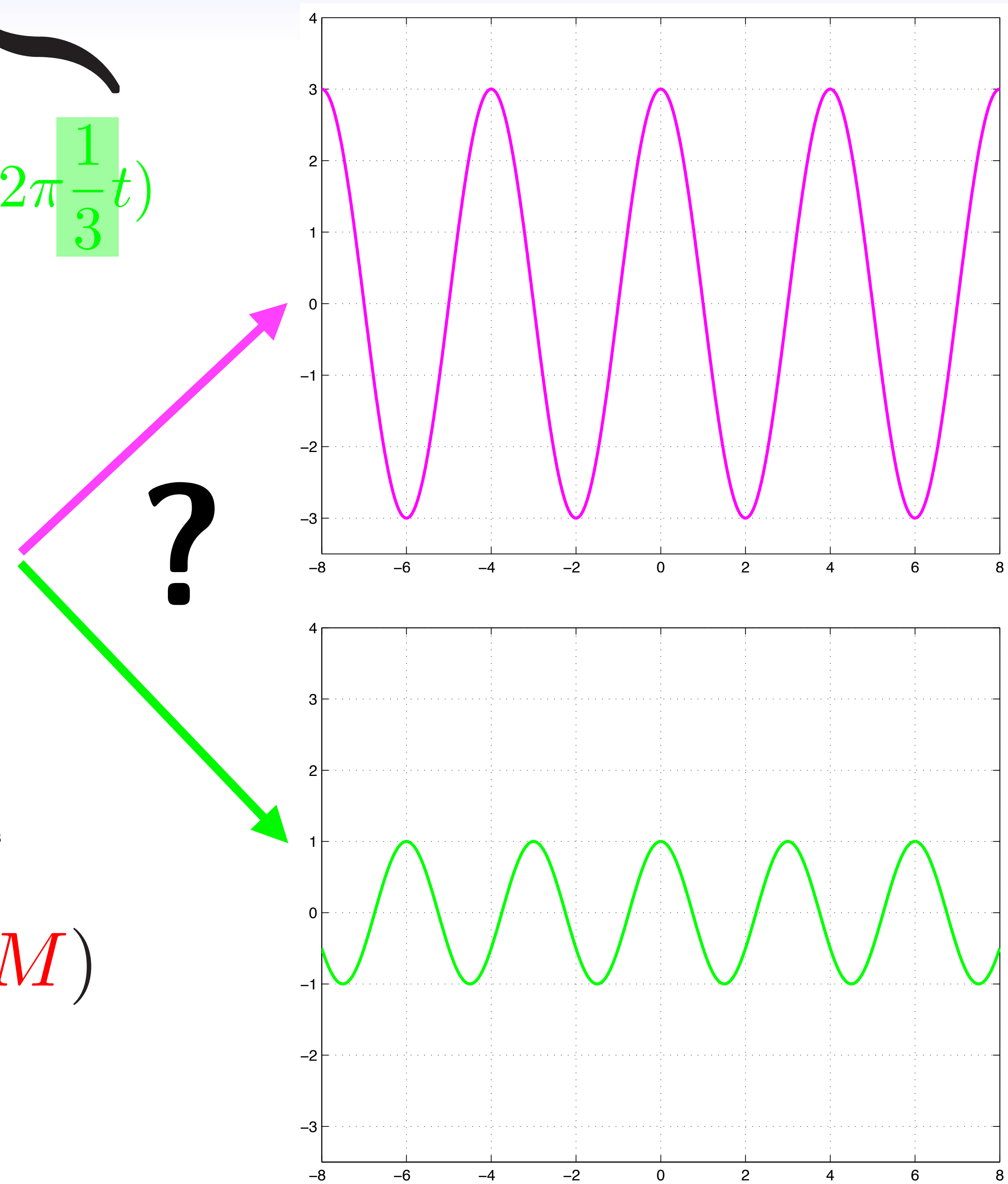
Spectral estimation

$$x(t) = \underbrace{3 \cdot \exp(i2\pi \tfrac{1}{4} t)}_{x_1(t)} + \underbrace{1 \cdot \exp(i2\pi \tfrac{1}{3} t)}_{x_2(t)}$$





$$\{x_m\}_{m=-M}^{M} = x(-M : M)$$

$$M = 2$$

Prony method

samples *parameters*

- Given
$$x_m = \sum_{k=1}^{K} \rho_k \big(\underbrace{e^{-j\omega_k}}_{z_k}\big)^m$$
$$\forall m = -M, \ldots, M$$
recover
$$\rho_k \in \mathbb{C}$$
$$\omega_k \in [-\pi, \pi]$$

- **Annihilating filter:** $H(z) = \prod_{k=1}^{K} (z - \overline{z_k}) = \sum_{k=0}^{K} h_k z^k$

$$\sum_{j=0}^{K} h_j x_{m-j} = \sum_{j=0}^{K} h_j \left(\sum_{k=1}^{K} \rho_k z_k^{m-j} \right) = \sum_{k=1}^{K} \rho_k z_k^m \underbrace{\left(\sum_{j=0}^{K} h_j z_k^{-j} \right)}_{H(\overline{z_k}) = 0} = 0$$

Prony method

- $$\sum_{j=0}^{K} h_j x_{m-j} = 0, \quad \forall m = -M+K, \ldots, M$$

$$\iff$$

$$\begin{pmatrix} x_{-M+K} & \cdots & x_{-M} \\ \vdots & \ddots & \vdots \\ x_M & \cdots & x_{M-K} \end{pmatrix} \begin{pmatrix} h_0 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\underbrace{}_{T_K} \qquad \underbrace{}_{h}$$

- $$\boxed{x * h = 0 \iff T_K h = 0}$$

Prony method

- $$\omega_k = \arg(\overline{z_k})$$

- Roots of $H(z) = \prod_{k=1}^{K} (z - \overline{z_k}) = \sum_{k=0}^{K} h_k z^k$

- $T_K h = 0 \Rightarrow h =$ Right singular vector associated to the singular value zero

Prony method

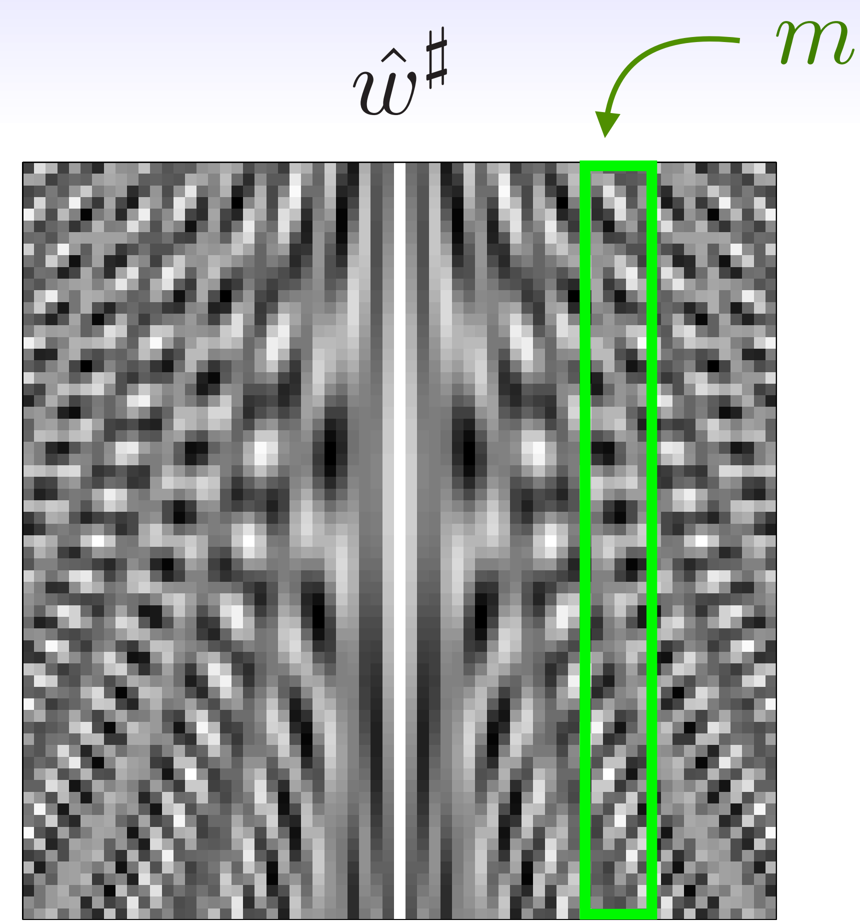
- $$x_m = \sum_{k=1}^{K} \rho_k \left(e^{-j\omega_k}\right)^m \quad \forall m = -M, \dots, M \iff x = U\rho$$

- with $U = \begin{pmatrix} e^{jM\omega_1} & \cdots & e^{jM\omega_K} \\ \vdots & \ddots & \vdots \\ e^{-jM\omega_1} & \cdots & e^{-jM\omega_K} \end{pmatrix}$ size $(2M+1) \times K$

- Solve $U^H U \rho = U^H x \iff \rho = (U^H U)^{-1} U^H x$

Application of the Prony method

- $\theta_k = \mathbb{E}[\theta_{m,k}] \qquad \alpha_k = \mathbb{E}[\alpha_{m,k}]$
- $\alpha_{m,k} = |d_{m,k}| \cos(\theta_{m,k})$
- $\theta_{m,k} = \arctan(W f_{m,k}/m)$
- **Prony** $\longrightarrow \{d_{m,k}\}_{m,k} \;\; \{f_{m,k}\}_{m,k}$
- $\hat{w}^\sharp_m = \hat{w}^\sharp[m,:] = \sum_{k=1}^{K} d_{m,k} a(f_{m,k})^T$, with $d_{m,k} \rightarrow c_k e^{i\phi_{m,k}}$
- $\hat{w}^\sharp_m = \hat{w}^\sharp[m,:] = \sum_{k=1}^{K} c_k a(f_{m,k}, \phi_{m,k})^T$



$$c_k = \frac{\alpha_k}{\cos\theta_k}$$

$$f_{m,k} = \frac{\tan(\theta_k)m}{W}$$

$$\phi_{m,k} = -\frac{2\pi\eta_k m}{W}$$

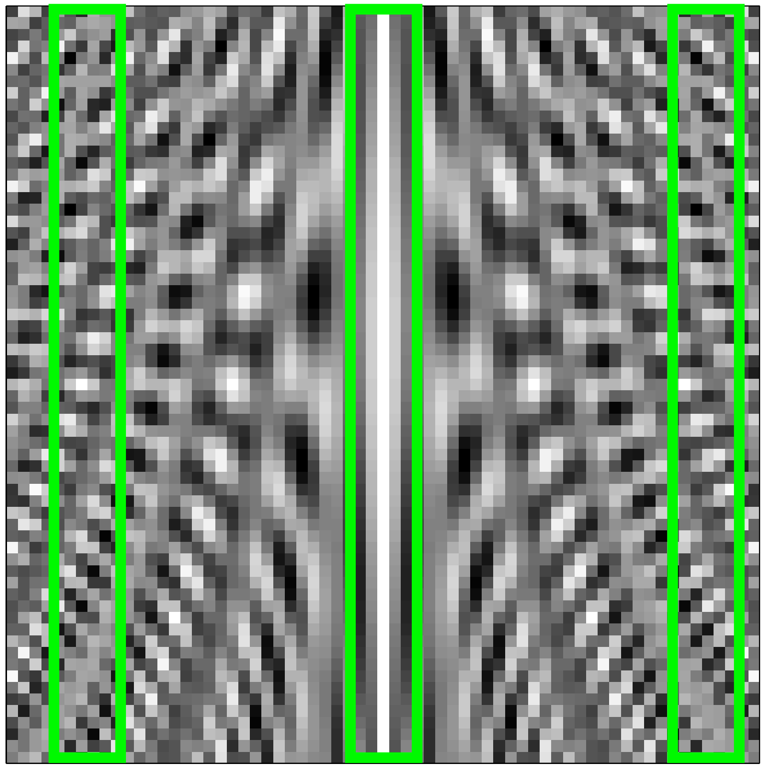
Application of the Prony method

$$\phi_{m,k} = -\frac{2\pi\eta_k m}{W}$$

$e^{i\phi_{1,k}} \cdots e^{i\phi_{M,k}} \cdots e^{i\phi_{W,k}}$ **Prony** → η_k

$\div c_k$

$d_{1,k} \cdots d_{M,k} \cdots d_{W,k}$



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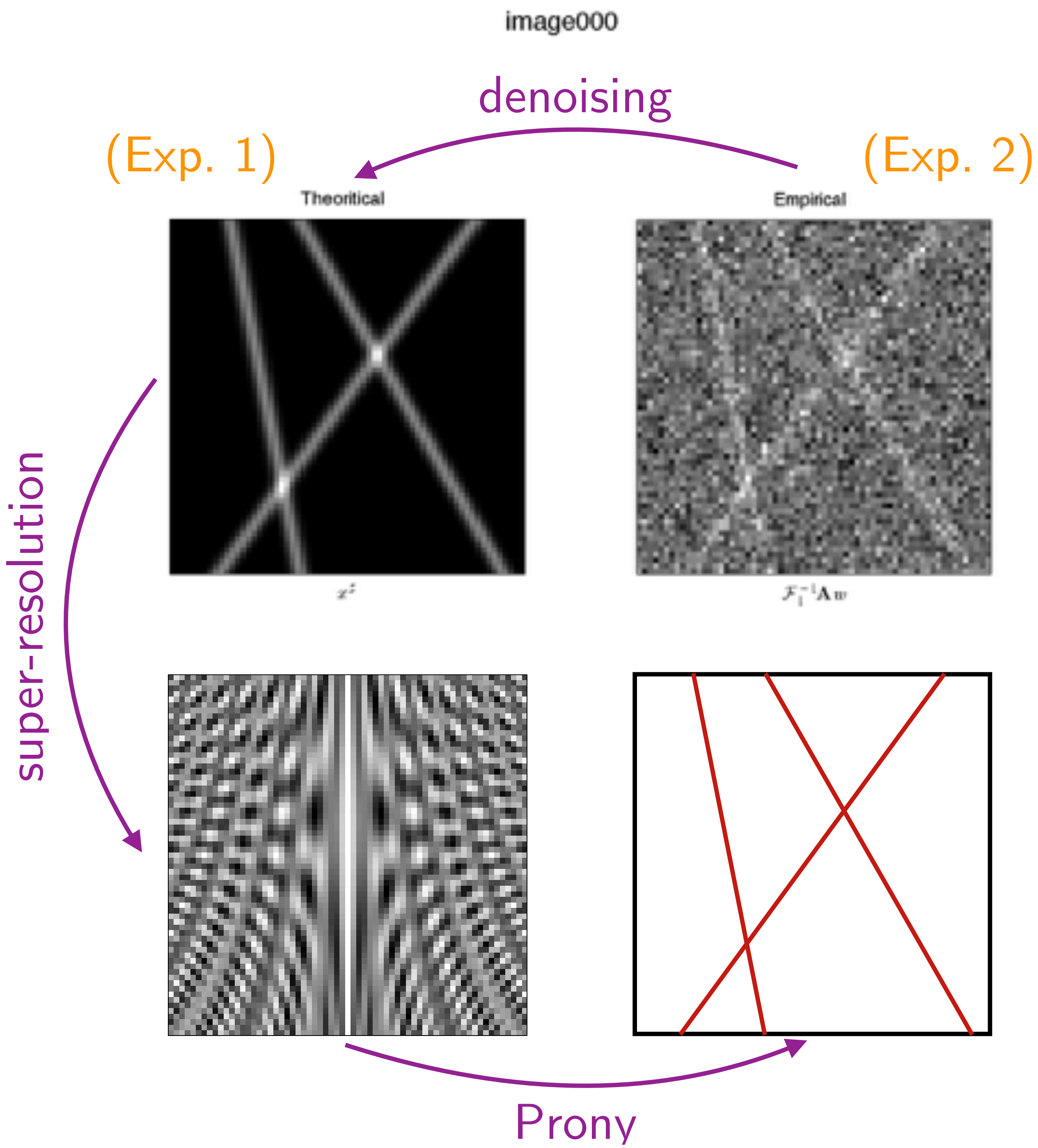
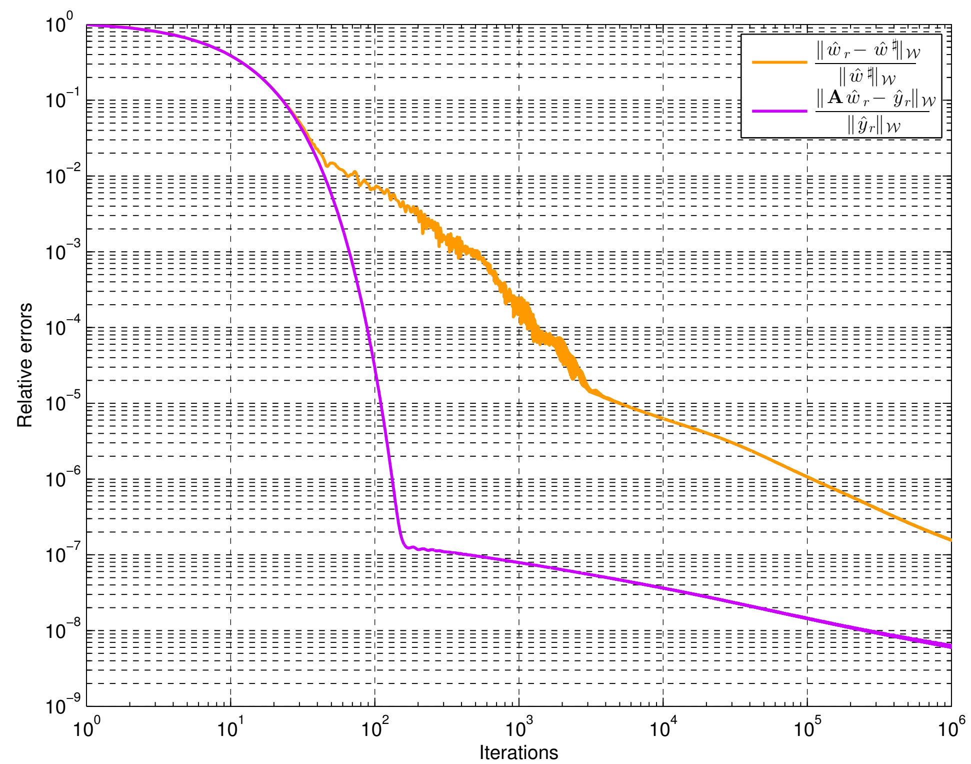
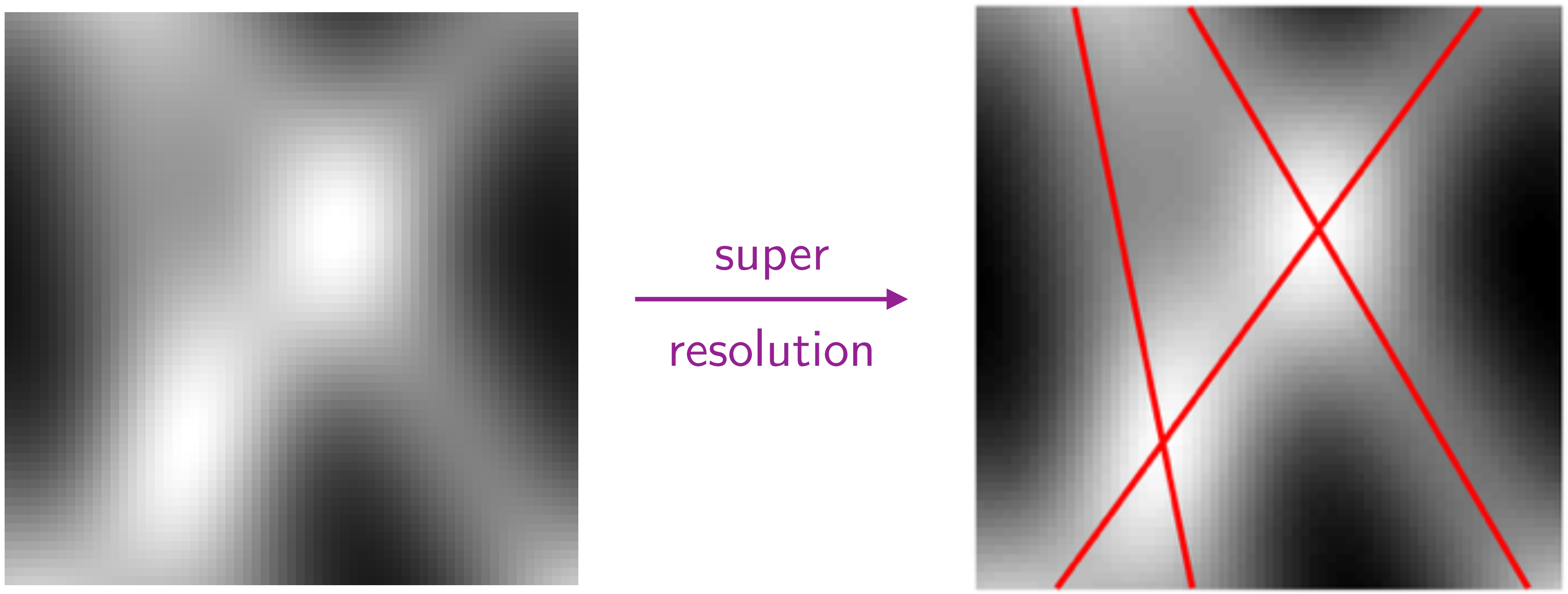
Numerical experiments





	Experiment 1	Experiment 2
Δ_θ/θ	$(10^{-7}, 3.10^{-6}, 7.10^{-7})$	$(10^{-2}, 6.10^{-2}, 9.10^{-2})$
Δ_α/α	$(10^{-7}, 10^{-7}, 10^{-7})$	$(10^{-2}, 9.10^{-2}, 2.10^{-1})$
Δ_η	$(4.10^{-6}, 7.10^{-6}, 7.10^{-6})$	$(5.10^{-2}, 4.10^{-2}, 3.10^{-2})$

Numerical experiments



	Experiment 3
Δ_θ/θ	$(6.10^{-7}, 9.10^{-5}, 8.10^{-6})$
Δ_α/α	$(4.10^{-5}, 2.10^{-5}, 2.10^{-5})$
Δ_η	$(5.10^{-5}, 10^{-4}, 3.10^{-4})$

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Conclusion and future work

Conclusion

- Model of lines super-resolution based on the atomic norm.
- Solve the optimization problem by primal-dual algorithm.
- Estimate the line parameters by Prony method.

Future work

- Extend the model to every angle range.
- Application to inpainting problems.
- Use local patch to deal with curves super-resolution.

Bibliography

Selected papers

- E. J. Candès, and C. Fernandez-Granda, ''Toward a mathematical theory of super-resolution'', *Communications on Pure and Applied Mathematics*, 2014.
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Questions ?

Thank you for your attention.